Dictionary of Selected Astrophysical Equations

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Preface

The purpose of this text is to present a selection of common astrophysical equations, illustrated with examples and solutions. Some textbooks on the subject present a number of equations without examples, in some cases neglecting to explain the terms contained in the equation or even to state which system of units is to be used. It is hoped that students will find many of those equations listed here, with terms clearly described and with clear examples and solutions given.

The text begins with a Prologue, which contains a number of tables of information useful to an astrophysics student.

The Prologue is followed by Part I, which contains alphabetically arranged sets of basic equations, with examples and solutions. The examples in this section use SI units unless otherwise stated.

Part II contains a set of PDF versions of Maple™ worksheets. These illustrate extended problems, usually involving more than a single equation. Both SI and cgs units are used, depending on the problem and the relevant equations.

An Appendix contains a list of common abbreviations found in astrophysical texts.

Entries in the Table of Contents are electronically linked to their respective pages for ease of use.

Equations and problems in Part I appear without source citations. It is felt that the equations are in the public domain, and examples that I have not created myself have appeared in at least two published texts without sources.

Source citations are included in all Part II examples that I did not create myself.

Please contact me if I have missed or incorrectly stated a source (robleerose@yahoo.ca).

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	EQUATION	Part 1	Part 2
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Part I

An Alphabetical List of Common Astrophysical Equations with Examples

A

Absolute magnitude

$$M = m - 5\ log_{10}(d) + 5$$

m = apparent magnitude
d = distance in parsecs

Example: The star Fomalhaut is 7.7 parsecs distant. Its apparent magnitude is 1.16. Therefore, its absolute magnitude is 1.72.

$$1.16 - 5\ log_{10}(7.7) + 5 = 1.72$$

Acceleration of a continuously varying mass

$$a = \frac{1}{m} \times I$$

$$I = v_e \times \frac{m_l}{t}$$

m = mass
I = impulse (force)
v_e = constant velocity of escaping mass
m_l/t = rate of mass loss

Example: A fully loaded Saturn V rocket has a mass of $2.80 \times 10^6\ kg$. The rate at which the exhaust leaves the rocket is $2.40 \times 10^3\ m\ s^{-1}$, and the resulting mass loss rate is $1.40 \times 10^4\ kg\ s^{-1}$. Therefore, the initial acceleration, minus the gravitational acceleration of the Earth, is $2.2\ m\ s^{-2}$.

$$\frac{1}{(2.80 \times 10^6)}(2.40 \times 10^3)(1.40 \times 10^4) - 9.8 = 2.2$$

Add two magnitudes

$$m_{total} = 2.5 * log_{10}\left(10^{-m_1*0.4} + 10^{-m_2*0.4}\right)$$

$m_1, m_2 =$ magnitudes

Example: Two stars with magnitudes 5.1 and 4.6 are too close together in the sky to be resolved by the unaided eye. Their combined magnitude is 4.07.

$$2.5 * log_{10}\left(10^{-5.1*0.4} + 10^{-4.6*0.4}\right) = 4.07$$

Adiabatic sound speed

$$v_s = \left(\frac{5P_{av}}{3\rho_{av}}\right)^{\frac{1}{2}}$$

P_{av} = average pressure
ρ_{av} = average density

Example: In the case of a monatomic gas, the solar adiabatic sound speed, with average pressure = $1.35 * 10^{14} Nm^{-2}$ and average density = $1410\, kg\ m^{-3}$ is $4 * 10^5 [m\ s^{-1}]$.

$$\left(\frac{5 * 1.35 * 10^{14}}{3 * 1410}\right)^{\frac{1}{2}} = 4 * 10^5 [m\ s^{-1}]$$

Angular distance on sphere

$$\Psi = \arccos(\sin\theta_1 \sin\theta_2 + \cos\theta_1 \cos\theta_2 \cos(\phi_1 - \phi_2))$$

$\phi_1, \phi_2 =$ right ascension, two points
$\theta_1, \theta_2 =$ declination, two points

Example: Right ascension and declination of Alpha Centauri and Proxima Centauri in radians are, respectively 3.838, -1.062 and 3.795, -1.094. Their separation on the sky is 0.0379 radians or 2.173 degrees.

$$\arccos(\sin(1.094)\sin(1.062) + \cos(1.094)\cos(1.062) \times \cos(3.838 - 3.795)) = 0.0379$$

Angular frequency

$$\omega = \frac{2\pi}{P} = 2\pi\nu = \frac{v}{r}$$

P = period
ν = frequency
v = tangential velocity
r = radius

Example: A pulsar discovered in 2005 has a spin period of 1.4 ms. Therefore, its angular frequency in radians per second is 4487.99. Divide this figure by 2π to get the angular frequency in Hz, 714.29 Hz.

$$\frac{2\pi}{1.4 \times 10^{-3}} = 4487.99$$

Angular momentum

$$L = I\omega$$
$$L = rmv$$

I = moment of inertia
ω = angular frequency
r = radius of rotation
m = mass
v = tangential velocity

Example: The Crab pulsar has a mass of $2.784 \times 10^{30} kg$, a radius of 10^4 m, and a period of 33.5029×10^{-3}s. Estimate its **moment of inertia** (*I*) as that of a homogeneous sphere ($I = \frac{2}{5}MR^2$) to find its angular momentum.

$$\frac{2}{5} \times (2.784 \times 10^{30}) \times (10^4)^2 \times \frac{2\pi}{33.5029 \times 10^{-3}} = 2.088 \times 10^4 [kg\ m^2\ s^{-1}]$$

Average energy generation rate $\epsilon \approx \frac{L}{M}$

L = luminosity
M = mass

Example: The mass of the Sun is 1.99×10^{30} kg. Its luminosity is $3.827 \times 10^{26}\ W$. Therefore, the average energy generation rate of the whole Sun is approximately $1.92 \times 10^4\ W/kg (= Js^{-1}kg^{-1})$.)

$$\frac{3.827 \times 10^{26}\ W}{1.99 * 10^{30}\ kg} \approx 1.92 \times 10^4\ [W/kg]$$

Average Intensity

$$J_\nu(z) = \frac{1}{2}\int_{-1}^{1} I_\nu(z, u) du$$

I_ν = monochromatic intensity
u = cosine of angle of incidence
Z = vertical depth

Example: In the Eddington approximation, the specific intensity can be expressed in the form $I_\nu(\tau, u) = a(\tau) + u \times b(\tau)$ where τ is the optical depth. In this case, $J_\nu = a(\tau)$.

$$\frac{1}{2}\int_{-1}^{1}(a(\tau) + u \times b(\tau))du = a(\tau)$$

If $\tau \gg 1$, then the average intensity is equal to the Planck function, B_ν.

B

Balmer Rydberg equation for hydrogen

$$\frac{1}{\lambda} = R \times \left(\frac{1}{n_f^2} - \frac{1}{n_i^2}\right)$$

λ = wavelength

$n_f =$ final energy level (1 for Lyman series; 2 for Balmer series; 3 for Paschen series; 4 for Pfund series)

$n_i =$ Initial energy level ($= n_f +$ any positive integer)

R = Rydberg constant ($1.0973732 \times 10^7 \, m^{-1}$)

Example: If the hydrogen electron jumps from level 5 to level 3 (Paschen series), it emits a photon with wavelength 1281.5 nm.

$$\frac{1}{\lambda} = 1.0973732 \times 10^7 \times \left(\frac{1}{5^2} - \frac{1}{3^2}\right) \Longrightarrow \lambda = 1281.5 \, nm$$

Barometric equation

$$P(h) = P(h_0) \; exp(-Mgh/RT)$$

P(h) = pressure at height *h*
$P(h_0)$ *= pressure at surface*
M = molar mass of the gas
g = gravitational acceleration
R = universal gas constant
T = temperature
h = height

Example: Plug in the appropriate values for the Earth to obtain a form of the equation giving pressure in relation to height above the surface.

$$P(h) = P(h_0)\,[kPa]\,exp\left(-\frac{0.02896\,[kg\,mol^{-1}]\,9.807\,[m\,s^{-2}]\,h\,[m]}{8.3143\,\left[\frac{Nm}{mol{\cdot}K}\right]\,288.15\,[K]}\right) = 101.325\,exp(-0.000119\,h)\,[kPa]$$

Binary star mass ratio

$$\frac{m_B}{m_A} = \frac{a_A}{a_B}$$

m_A = mass of star A
m_B = mass of star B
a_A = distance of star A from barycentre
a_B = distance of star B from barycentre

Example: Alpha Centauri A and Alpha Centauri B revolve around a common barycentre. The ratio of their distances from the barycentre is 0.825. If the mass of Alpha Centauri A is 1.1 $M_\odot$, then it follows that the mass of Alpha Centauri B is 0.907 $M_\odot$.

$$m_B = 0.825 \times 1.1 = 0.907$$

Binary star semi-major axis

$$a_B = a\left(1 + \frac{a_A}{a_B}\right)^{-1}$$

a = semi-major axis
a_A = distance of star A from barycentre
a_B = distance of star B from barycentre

Example: In the Sirius binary star system, Sirius A is 6.43 AU from the barycentre, and Sirius B is 13.4 AU from the barycentre. Therefore, the semi-major axis of the system is 9.05 AU.

$$13.4 = a\left(1 + \frac{6.43}{13.4}\right)^{-1} \implies a = 9.05\,AU$$

Black hole mass in quasar core

$$M_{BH}(MgII) = 10^{6.86}\left(\frac{FWHM_{line}}{10^3\,km\,s^{-1}}\right)^2\left(\frac{\lambda L_\lambda(3000\text{Å})\,erg\,s^{-1}}{10^{44}\,erg\,s^{-1}}\right)^{0.5} M_{Sun}$$

$FWHM_{line}$ = full width at half maximum of the spectral line
λL_λ = measured luminosity of quasar
M_{sun} = mass of Sun

Example: An empirical equation devised by De Rosa et al (2014)[†] may be used to estimate the mass of a black hole located in the centre of a quasar, by means of the broadening of the 3000Å line of MgII. Measurements of this line in the spectrum of the quasar J2348-3054, one of the most distant objects in the visible Universe, gives a result of 5446 km/s. The measured luminosity of this quasar is $0.94 \times 10^{46}\,erg\ s^{-1}$. It follows that the mass of the black hole contained in this quasar is approximately equal to the mass of two billion Suns.

† De Rosa, G., et al. (2014). Black hole mass estimates and emission-line properties of a sample of redshift z > 6.5 quasars. *APJ*, 790. 145.

$$10^{6.86} \times \left(\frac{5446}{10^3}\right)^2 \times \left(\frac{0.94 \times 10^{46}}{10^{44}}\right)^{.5} = 2.08 \times 10^9\,M_{sun}$$

Bolometric correction

$$m_{bol} = V + BC$$

m_{bol} = apparent bolometric magnitude
V = apparent visual magnitude
BC = bolometric correction

Example: The apparent visual magnitude of Sirius is -1.44. Its bolometric correction is -0.09. Therefore, its apparent bolometric magnitude is -1.53.

$$m_{bol} = -1.44 + (-0.09) = -1.53$$

Bolometric equation

$$M - M_{sun} = -2.5 \ log\left(\frac{L}{L_\odot}\right)$$

M = absolute magnitude of star
M_{sun} = absolute solar magnitude (4.74)
L = luminosity of star
$L_\odot$ = solar luminosity

Example: The star Sirius has an absolute magnitude of 1.36. The absolute magnitude of the Sun is 4.74. Therefore, Sirius is 22.5 times as luminous as the Sun.

$$1.36 - 4.74 = -2.5 \ log\left(\frac{L}{L_\odot}\right) \Longrightarrow \left(\frac{L}{L_\odot}\right) = 22.5$$

C

Central pressure of gravitationally bound sphere

$$P > \frac{3M^2G}{8\pi R^4}$$

M = mass
G = gravitational constant
R = radius

Example: A rough estimate of a lower bound on the central pressure of the Sun can be found by substituting the solar values for mass and radius into the equation. The result is $1.35 \times 10^{14}\ Pa$. The actual pressure is $2.5 \times 10^{16}\ Pa$.

$$\frac{3 \times (1.99 \times 10^{30})^2 \times 6.67 \times 10^{-11}}{8 \times (6.96 \times 10^8)^4} = 1.35 \times 10^{14}\ Pa$$

Central temperature

$$T_c = \frac{\xi_1}{4y_1}\mu(1-\beta)\frac{GM_* m_p}{RK}$$

$\xi_1 = 6.897$
$y_1(\xi_1) = 2.018$
$\mu = 0.68$
$\beta =$ contribution of radiation pressure to total pressure
G = gravitational constant
$M_* =$ mass of star
$m_p =$ mass of proton
$R_* =$ radius of star
k = Boltzmann constant

Example: This formula is derived from the **Lane-Emden equation** of order 3, which approximates stars such as the Sun. Inserting solar values yields a central temperature of $1.34 \times 10^7\, K$, which is reasonably close to the currently accepted figure of $1.58 \times 10^7\, K$, derived from solar models.

$$\frac{6.897}{4 \times 2.018} \times 0.68(1 - 6.58 \times 10^{-4}) \times \frac{6.67 \times 10^{-11} \times 1.989 \times 10^{30} \times 1.67 \times 10^{-27}}{6.96 \times 10^8} \times 1.38 \times 10^{-23} = 1.34 \times 10^7$$

Centripetal acceleration

$$a_c = \frac{v^2}{r}$$

v = velocity
r = distance to the centre

Example: The Earth's average velocity in its orbit around the Sun is 2.99×10^4 m/s. Its distance from the Sun is approximately 1.50×10^{11} m. Therefore, its centripetal acceleration is $5.96 \times 10^{-3}\, m\, s^{-2}$.

$$\frac{(2.99 * 10^4)^2}{1.50 * 10^{11}} = 5.96 * 10^{-3}$$

Colour index temperature

$$T = \frac{8540}{(M_B - M_V) + 0.865}$$

$M_B =$ magnitude, blue filter
$M_V =$ magnitude, visual filter

Example: If a star has an absolute visual magnitude of 1.95 and an absolute blue filter magnitude of 2.10, the temperature of the surface is approximately 8,400 K.

$$\frac{8540}{(2.10 - 1.95) + 0.865} = 8{,}414$$

Column density for optically thin gas

$$\left[\frac{N_{HI}}{cm^{-2}}\right] = (1.82 \times 10^{18}) \int_{\nu} \left[\frac{T_B(\nu)}{K}\right] \left[\frac{dv}{km\ s^{-1}}\right]$$

$T_B =$ brightness temperature
$\nu =$ frequency

Example: The optically thin HI line emission from the galaxy NGC 2903 is spread over a solid angle of $\Omega_s = 1.8 \times 10^{-5}\, sr$. The intensity measured at the radio telescope is $\int I_\nu\, dv = 0.0458\, Jy\ beam^{-1}\, km\ s^{-1}$. The beam solid angle is $\Omega_b = 5.21 \times 10^{-9}\, sr$. Converting from janskys and dividing by the beam solid angle gives $\int I_\nu\, dv = 8.79 \times 10^{-17}\, erg\, s^{-1}\, cm^{-2}\, km\ s^{-1}$. Converting this amount to a brightness temperature using $T_B = \frac{\lambda^2}{2 \times k} I_\nu$, where k is the Boltzmann constant, and λ is 21.1 cm for HI, gives $T_B(\nu) = 142\, K\, km\ s^{-1}$. Inserting this figure in the equation for column density gives gives a column density for Hydrogen I in NGC 2903 of $2.58 \times 10^{20}\, cm^{-2}$.

$$\left[\frac{N_{HI}}{cm^{-2}}\right] = (1.82 \times 10^{18}) \int_{\nu} \left[\frac{142\, K\, km\, s^{-2}}{K}\right] \left[\frac{dv}{km\, s^{-1}}\right] = 2.58 \times 10^{20}\, cm^{-2}$$

Coulomb barrier

$$F = k_e \frac{Z_1 Z_2}{r} e^2$$

$$k_e = \frac{1}{4\pi\epsilon_0} = 8.9875 \times 10^9\, N\, m^2\, C^{-2}$$

(Coulomb's constant) where

$$\epsilon_0 = \frac{1}{4\pi\,(8.9875 \times 10^9)} = 8.85 \times 10^{-12}\, C^2\, N^{-1}\, m^{-2}$$

(permittivity of free space)

$Z_1, Z_2 =$ atomic number
$r =$ distance between point charges
$e =$ elementary charge

Example: In the core of a Main Sequence star, two hydrogen nuclei (protons) may come within a distance of $10^{-15}\, m$ of each other. In order to fuse, they must overcome a Coulomb barrier of $2 \times 10^{-13}\, J$. Classically, this would be impossible at the temperatures in the cores of such stars, but a few may overcome the Coulomb barrier by means of "quantum tunnelling". (This formula is derived from **Coulomb's force law**.)

$$(8.9875 \times 10^9) \frac{1 \times 1}{10^{-15}} (1.60 \times 10^{-19})^2 = 2 \times 10^{-13}$$

Coulomb's force law

$$F = k_e \frac{Q_1 Q_2}{r^2}$$

$$k_e = \frac{1}{4\pi\epsilon_0} = 8.9875 \times 10^9\, N\, m^2\, C^{-2}$$

(Coulomb's constant) where

$$\epsilon_0 = \frac{1}{4\pi\,(8.9875 \times 10^9)} = 8.85 \times 10^{-12}\, C^2\, N^{-1}\, m^{-2}$$

(permittivity of free space)

$Q_1, Q_2 =$ charges
$r =$ distance between charges

Example: In a hydrogen atom, the electron ($Q = -\,1.60 \times 10^{-19}\, C$) and proton ($Q = +\,1.60 \times 10^{-19}\, C$) are separated by a distance of $5.3 \times 10^{-11}\, m$. Therefore, the force between them is $8.2 \times 10^{-8}\, N$, and is attractive. (The Coulomb force law provides the basis for the **electric field vector for point charge** and for the **Coulomb barrier**.)

$$(8.9875 \times 10^9)\left(\frac{(-1.6 \times 10^{-19})(1.6 \times 10^-19)}{(5.3 \times 10^{-11})^2}\right) = -\,8.2 \times 10^{-8}$$

Critical density of the Universe

$$\rho_c = \frac{3H_0^2}{8\pi G}$$

$H_0 =$ the Hubble constant
G = gravitational constant

Example: The currently accepted value of the Hubble constant is $71\, km\, s^{-1}\, Mpc^{-1} = 2.30 \times 10^{-18}\, s^{-1}$. Therefore, the critical density is $9.46 \times 10^{-27}\, kg\, m^{-3}$. Baryonic matter accounts for only about 4% of this amount.

$$\frac{3 \times (2.3 * 10^{-18})^2}{8 \times \pi \times 6.67408 * 10^{-11}} = 9.46 \times 10^{-27}$$

Cyclotron frequency for electrons

$$\nu_c = \frac{eB}{2\pi m_e}$$

ν_c = frequency (MHz)
e = electron charge
B = magnetic field
m_e = electron mass

Example: When gas in a white dwarf binary spirals down on one of the stars, the intense magnetic field, typically 1000 T, gives rise to cyclotron radiation from the spiralling electrons. The resulting frequency in this case is 2.3×10^{13} Hz.

$$\frac{1.602 * 10^{-19} \times 1000}{2 \times \pi \times 9.109 * 10^{-31}} = 2.8 \times 10^{13}\, Hz$$

D

D-sigma relation for elliptical galaxies

$$\log_{10} D = 1.333\ \log_{10} \sigma + C$$

$D =$ galaxy's angular diameter (inversely proportional to distance *d*
$\sigma =$ velocity dispersion
$C =$ constant determined by observation

Example: For the Virgo cluster, *C* = -1.237. For the Coma cluster, C = -1.967. Therefore, the Coma cluster is more than five times as far away as the Virgo cluster.

$$\frac{d_2}{d_1} = \frac{D_1}{D_2} = 10^{C_1 - C_2} = 10^{-1.237-(-1.967)} = 5.37$$

De Broglie wavelength

$$\lambda = \frac{h}{\sqrt{2mE}}$$

h = Planck's constant
m = mass
E = energy

Example: Since $E = \frac{3}{2}kT$ for a gas, the equation may be written as $\lambda = h/\sqrt{3mkT}$. To find the de Broglie wavelength of helium at 300 K, note that the mass of the helium atom is $6.64 \times 10^{-27}\,kg$. Therefore, its de Broglie wavelength is $7.27 \times 10^{-11}\,m$.

$$\frac{6.6 \times 10^{-34}}{\sqrt{3 \times 6.64 \times 10^{-27} \times 1.38 \times 10^{-23} \times 300}} = 7.27 \times 10^{-11}\, m$$

De Vaucouleurs' formula (for elliptical galaxies)

$$I(r) = I(r_e)\exp\left\{-7.669\left[\left(\frac{r}{r_e}\right)^{\frac{1}{4}} - 1\right]\right\}$$

r = radial distance from centre
r_e = distance of half intensity
I = intensity
$I(r_e)$ = Intensity at r_e.

Example: This empirical formula (sometimes mistakenly termed a "law") gives an intensity profile of an elliptical galaxy whose distance of half intensity from the centre is known. A more general form of the formula, known as Sersic's law, replaces the 1/4 exponent with 1/n, where n can vary from 1 to 4 depending on the galaxy, and the factor in the exponent must be changed accordingly.

Several formulas can be derived from de Vaucouleurs' formula. One of these is

$$m = r_e - 5 * log_{10}(a) - 12.7$$

which provides an estimate of the apparent magnitude (*m*) if the light scale-length (*a*) is known. For instance, the Virgo Cluster galaxy VCC 753 has an r_e value of 24.4 and a light scale-length of 0.15", giving an apparent magnitude of 15.82. If this figure differs from the measured magnitude, the difference is likely due to extinction. Since the distance of this galaxy is approximately 17.6 Mpc, the **distance-modulus** formula gives an absolute magnitude of -15.43. With this figure, the **luminosity** formula

$$L = 10^{(M_{sun} - M_{galaxy})/2.5)}$$

gives a luminosity for this galaxy of $1.3 \times 10^8\, L_{sun}$.

Differential tidal forces

$$dF = -\left(\frac{2GMm}{R^3}\right) dR$$

G = gravitational constant
M = mass of larger body
m = mass of smaller body
R = separation distance

Example: To use this equation to determine the relative effects of the Sun and the Moon on the Earth's tides, write the equation as

$$dF = \frac{mass\ of\ Sun}{mass\ of\ Moon} \times \left(\frac{distance\ of\ Moon}{distance\ of\ Sun}\right)^3$$

and insert the corresponding values. The result shows that the Sun provides less than half of the tidal force provided by the Moon.

$$\frac{1.99 \times 10^{30}}{7.36 \times 10^{22}} \times \left(\frac{3.84 \times 10^5}{1.50 \times 10^8}\right)^3 = 0.45$$

Distance modulus

$$M - m = 5 - 5 \log d - A$$

M = absolute magnitude
m = visual magnitude
d = distance (pc)
A = extinction in magnitude

Example: The star Sirius has a visual magnitude of -1.47 and is at a distance of 2.64 pc. Extinction over this distance is essentially zero. Therefore, the absolute magnitude of Sirius is +1.42.

$$-1.47 + 5 - 5 \times log_{10}(2.64) - 0 = 1.42$$

Distance modulus (bolometric) $M_{bol} = m_v + 5 - 5\log d - A_v - BC$

M_{bol} = absolute bolometric magnitude
m_v = visual magnitude
d = distance (pc)
A_v = extinction in magnitude
BC = bolometric correction

Example: The star Rigel has an apparent visual magnitude of 0.12. It is at a distance of 236.9 parsecs. The magnitude extinction is 0.1128, and the **bolometric correction** is 0.66. Rigel's absolute bolometric magnitude is, therefore, -7.3.

$$0.12 + 5 - 5\log(236.9) - 0.1128 - 0.66 = -7.3$$

Distance now

$$d_{now} = \left(\frac{c}{H_0}\right) ln(1 + z)$$

d_{now} = distance now
H_o = Hubble constant
z = redshift
c = speed of light

Example: The quasar PC1247+346 was at a distance of approximately 3940 Mpc, when its observed, red-shifted (z = 4.897) light was emitted, In the roughly 12.9×10^9 years since that light was emitted, the Universe has expanded, and, assuming that the value of the Hubble constant is 68 km\s\Mpc (= 68000 m/s/Mpc), the quasar is now at a distance of 7823 Mpc.

$$\left(\frac{2.99792458 \times 10^8}{68 \times 10^3}\right) \times \ln(1 + 4897) = 7823$$

Doppler broadening from rotation

$$\Delta\lambda = \frac{FWHM}{2.35}$$

$$v = \frac{\Delta\lambda}{\lambda} \times c$$

$\Delta\lambda =$ Doppler broadening
$\lambda =$ wavelength of line
FWHM = full width at half maximum
v = rotational velocity
c = speed of light

Example: The B2V star HD145482 shows a broadened line at 492 nm. The FWHM of this line is 0.5 nm. This indicates a rotational velocity of 130 km/s.

$$\frac{0.5}{2.35} = 0.21$$

$$\frac{0.21}{492} \times 3 \times 10^8 = 130\,km/s$$

Doppler recession velocity

$$\frac{v}{c} = \frac{(z+1)^2 - 1}{(z+1)^2 + 1}$$

z = Doppler shift (redshift)
$v =$ recession velocity
c = speed of light

Example: The Doppler shift of the quasar 3C-273 is measured from its spectrum to be 0.158. Therefore, v/c = 0.14565, meaning that its recession velocity is nearly 15% of the speed of light, or 43,400 km/s.

$$\frac{(0.158+1)^2-1}{(0.158+1)^2+1} = 0.14565$$

Doppler shift from rotation $\Delta\lambda/\lambda = 2v_{eq}/c = 4\pi R/Pc$

$\Delta\lambda =$ Doppler shift
$\lambda =$ midpoint wavelength
$v_{eq} =$ velocity at equator
c = speed of light
R = radius
P = period

Example: The galaxy UGC 2936 rotates such that the hydrogen Balmer alpha line, measured at the approaching edge, is at 664.0 nm. At the receding edge, it is at 665.0 nm. The rotational velocity of this galaxy is, therefore, 226 km/s.

$$\frac{1}{664.5} = \frac{2 \times v}{3 \times 10^8} \Longrightarrow v = 226\, km/s$$

E

Einstein's mass-energy equation

$$E = \gamma mc^2 = \gamma \left(m_{init} - m_{final}\right) c^2$$

γ = Lorenz factor
m_{init} = initial mass
m_{final} = final mass
c = speed of light

Example: In hydrogen fusion, four hydrogen atoms combine to form one helium atom. The mass defect in this reaction ($m_{init} - m_{final}$) is 4.8418×10^{-29} kg. The Lorentz factor here may be taken to be equal to 1. Therefore, the energy released is 4.3516×10^{-12} *J* or 27 *MeV.*

$$4.8418 \times 10^{-29} \times (2.9979 \times 10^8)^2 = 4.3516 \times 10^{-12}$$

Electric field vector for point charge

$$\overrightarrow{E} = \frac{1}{4\pi\epsilon} \times \frac{q}{r^2}\,\hat{r}$$

$\frac{1}{4\pi\epsilon}$ ($= k_e, Coulomb's\ law\ constant) = 9.0 \times 10^9\ N m^2 C^{-2}$
q = electrostatic charge in coulombs
r = distance to the charge

Example: The strength and direction of the electric field 2.81 cm on the left hand side of a 7.2 mC negative charge is 8.21×10^7 *N/C* (= $8.21 \times 10^7\ V m^{-2}$) to the right. (This formula is derived from **Coulomb's law**.)

$$\frac{(9.0 \times 10^9) \times (7.2 \times 10^{-6})}{(2.81 \times 10^{-2})^2} = 8.21 \times 10^7$$

Electrostatic potential

$$V = k\,\frac{q}{r}$$

$\frac{1}{4\pi\epsilon}\,(= k, Coulomb's\,law\,constant) = 9.0 \times 10^9\,N\,m^2\,C^{-2}$
q = point charge
r = distance to the point charge

Example: The electrostatic potential at a point midway between two point charges of $2.5\,\mu C$ each, separated by two metres, is $4.5 \times 10^4\,V$.

$$2\left(9 \times 10^9\,\frac{Nm^2}{C^2}\right)\frac{2.5 \times 10^{-6}\,C}{1\,m} = 4.5 \times 10^4\,V$$

Energy of electromagnetic radiation

$$E = h\nu = \frac{hc}{\lambda}$$

h = Planck's constant
ν = frequency
λ = wavelength

Example: A photon with a wavelength of 530 nm appears green. This photon has an energy of 3.75×10^{-19} joules or 2.34 eV.

$$6.626 \times 10^{-34} \times \frac{3 \times 10^8}{530.0 \times 10^{-9}} = 3.75 \times 10^{-19}\,J$$

Energy of electron in orbital n of the hydrogen atom

$$E_n = \frac{-R}{n^2}$$

R = Rydberg condstant
n = orbital: 1, 2, 3, etc.

Example: The electron energy in the second excited state of hydrogen is 5.45×10^{-19} J.

$$E_2 = \frac{-2.18 \times 10^{-18}}{2^2} = 5.45 \times 10^{-19}$$

Energy of photon emitted or absorbed by hydrogen atom

$$E_{photon} = \left| R\left(\frac{1}{n_f^2} - \frac{1}{n_i^2}\right) \right|$$

R = Rydberg constant
$n_i =$ initial orbital (n = 1, 2, 3, etc.)
$n_f =$ final orbital (n = 1, 2, 3, etc.)

Example: A hydrogen electron jumping from the n = 3 orbital to the n = 2 orbital emits a photon with energy 3.03×10^{-19} J. The equation $E = h\nu$ can then be used to find the frequency of this photon.

$$\left| -2.18 \times 10^{-18} \left(\frac{1}{2^2} - \frac{1}{3^2}\right) \right| = 3.03 \times 10^{-19}$$

Equilibrium temperature of Solar System planet

$$T_{eq} = T_{sun}(1-a)^{1/4}\sqrt{\frac{R_{sun}}{2D}}$$

T_{sun} = temperature of the Sun
a = albedo
R_{sun} = radius of Sun
D = distance of planet from Sun

Example: Venus, which has an albedo of 0.6, is at a distance of 0.72 AU from the Sun. The solar temperature is 5770 K, and the solar radius is 7×10^5 kilometres. It follows that the equilibrium temperature of Venus is 262 K. However, the surface temperature is 735 K, owing to the run-away greenhouse effect of the Venusian atmosphere.

$$5770 \times (0.4)^{1/4}\sqrt{\frac{7 \times 10^5 \times 10^3}{2 \times (0.72 \times 149.6 \times 10^9)}} = 262\,K$$

Equivalent width (for a line that is zero outside the rectangular width)

$$W_\lambda = \int A_\lambda d\lambda$$

A_λ = line depth
λ = wavelength

Example: A **line depth** of 1/3 in a rectangular line with a width of $4\mathring{A}$ translates to an equivalent width of $\frac{4}{3}\mathring{A}$.

$$\int_x^{x+4} \frac{d\lambda}{3} = \frac{4}{3}$$

Escape velocity

$$v = \sqrt{\left(\frac{2GM}{r}\right)}$$

G = gravitational constant
M = mass of planet, star, etc.
r = distance from centre of mass

Example: Mars has an average mass of 0.64171×10^{24} kg. Its radius is 3385 km. Therefore, the escape velocity at the Martian surface is 5030 m/s.

$$\sqrt{\frac{2 \times 6.67408 \times 10^{-11} \times 0.64171 \times 10^{24}}{3385 \times 10^{3}}} = 5030$$

F

Faber-Jackson relation for elliptical galaxies

$$\log_{10}\sigma = -0.1M_B + 0.2$$

σ = velocity dispersion
M_B = absolute bolometric magnitude

Example: The elliptical galaxy M32, with an apparent magnitude of +9.03, has a velocity dispersion of 60 km/s. Its absolute bolometric magnitude is thus -15.8. The **distance modulus** formula can then be used to calculate the approximate distance of this galaxy, 9.25×10^5 parsecs.

$$\log_{10}(60) = -0.1M_B + 0.2 \implies M_B = -15.8$$

$$-15.8 - 9.03 = 5 - 5\log d \implies d = 9.25 \times 10^5$$

Flux (Stefan-Boltzmann law)

$$F = \sigma T^4 = \frac{L}{4\pi R^2}$$

σ = Stefan-Boltzmann constant
T = temperature
L = luminosity
R = radius or distance from source

Example: The flux at the surface of Sirius (T = 9940 K) is $5.54 \times 10^8\, W m^{-2}$.

$$5.670 \times 10^{-8} \times (9940)^4 = 5.54 \times 10^8$$

Flux: monochromatic

$$F_\nu = 2\pi \int_{-1}^{+1} I_\nu u\, du$$

I_ν = specific intensity
u = cosine of angle between direction of radiation and normal to the surface

Example: The monochromatic flux is equal to π times the Planck function, B_ν.

$$F_\nu = 2\pi \int_{-1}^{+1} I_\nu u\, du = 2\pi \int_0^1 B_\nu u\, du = 2\pi B_\nu \left[\frac{u^2}{2}\right]_0^1 = \pi B_\nu$$

Flux: photon number
(wavelengths in Angstroms)

$$N \equiv \frac{F_\lambda(m = 0)\Delta\lambda}{E}$$

N = number of photons
$f_\lambda(m = 0)\Delta\lambda \equiv 3.7 \times 10^{-12} \Delta\lambda\; J\, m^{-2}\, s^{-1}\, \text{Å}^{-1}$
(specific flux defined)
E = energy per photon

Example: The range of a V filter is $4000\,\text{Å} - 7000\,\text{Å}$, which is close to the range of wavelengths visible to the human eye. Therefore, $\Delta\lambda = 7000\,\text{Å} - 4000\,\text{Å} = 3000\,\text{Å}$, so that the total specific flux is $1.11 \times 10^{-8}\, J\, m^{-2}\, s^{-1}$. Dividing this number by the energy of a mid-range photon, $5500\,\text{Å}$, namely $3.6 \times 10^{-19}\, J$, gives the number of photons in the V-filter range for a zero-magnitude star, namely 3×10^{10}. (This calculation is used in determining the **signal-to-noise ratio**.)

Determine the energy of a mid-range photon ($\lambda = 5500\,\mathring{A}$), using the formula for the **energy of electromagnetic radiation**:

$$E = \frac{(6.6261 \times 10^{-34})(3 \times 10^{8})}{550 \times 10^{-9}} = 3.6 \times 10^{-19}\, J/photon$$

Calculate the total specific flux:

$$f_\lambda(m = 0)\Delta\lambda \equiv 3.7 \times 10^{-12} \times 3000 = 1.11 \times 10^{-8}\, J\, m^{-2}\, s^{-1}$$

Divide this by the energy per photon to get the total number of photons in this wavelength range from a zero-magnitude star observed through a V filter:

$$\frac{1.11 \times 10^{-8}}{3.6 \times 10^{-19}} = 3 \times 10^{10}$$

Note that for each 5-magnitude increase, the number of photons decreases by a factor of 100 (See **magnitude to flux**):

Magnitude	Flux in number of photons $s^{-1}\, m^{-2}$
0	3×10^{10}
5	3×10^{8}
10	3×10^{6}
15	3×10^{4}
20	3×10^{2}
25	3
30	3×10^{-2}

Free-fall time

$$t_{ff} = \left(\frac{3\pi}{32G\rho_0}\right)^{\frac{1}{2}}$$

G = gravitational constant

ρ_0 = density before contraction

Example: The dense core of a typical interstellar hydrogen cloud has a density of about $3.0 \times 10^{-17}\,kg\,m^{-3}$. The free-fall collapse time of the core is, therefore, approximately 3.8×10^5 years.

$$\left(\frac{3 \times \pi}{32 \times 6.67 \times 10^{-11} \times 3.0 \times 10^{-17}}\right)^{\frac{1}{2}} = 3.84 \times 10^5\,yr$$

Free-free absorption coefficient (synchrotron radiation)

$$\alpha_\nu^{ff} = 3.7 \times 10^{-2}\,\frac{T^{-0.5} Z^2\;g_{ff}\,n_e{}^2}{\nu^3}\left(1 - \mathrm{e}^{-\frac{h\nu}{kT}}\right)$$

T = temperature
Z = atomic number
g_{ff} = Gaunt factor for free-free emission
n_e = particle density
ν = frequency
h = Planck's constant
k = Boltzmann's constant

Example: The hydrogen gas in the Orion nebula is at a temperature of approximately 8000 K. The electron particle density is about $6 \times 10^9\,m^{-3}$; the atomic number of hydrogen is 1, and the applicable Gaunt factor is approximately 1. The synchrotron (free-free) absorption coefficient as a function of frequency, multiplied by the line-of-sight thickness of the nebula (4.29 parsecs), gives the **optical depth** of the gas (τ) as a function of frequency.

$$\tau = (3.7 * 10^{-2} * 8000.0^{-1/2} * 1^2 * \nu^{-3}\left[1 - exp\left(\frac{-6.626 * 10^{-27}) * \nu}{(1.381 * 10^{-16}) * 8000}\right)\right] * 1 * (6 * 10^9)^2 * 4.29 * (3.086 * 10^{16})$$

$$= \frac{1.97 * 10^{33} * [1 - \exp(-5.997 * 10^{-15} * \nu]}{\nu^3}$$

Frequency-wavelength relation $\nu = \frac{c}{\lambda}$

ν = frequency
c = speed of light
λ = wavelength

Example: Green light with a wavelength of 550 nm has a frequency of $5.45 \times 10^{14}\ Hz$.

$$\frac{2.9979 \times 10^{8}}{550 \times 10^{-9}} = 5.45 \times 10^{14}$$

G

Gravitational acceleration

$$g = \frac{GM}{r^2}$$

G = gravitational constant
M = mass of gravitating body
r = distance from centre of gravitating body

Example: The mass of the Earth is 5.97219×10^{24} kg, and its radius is 6.3781366×10^{6} m. Therefore, the gravitational acceleration at the surface is $9.8\, m\ s^{-2}$ downward.

$$-\frac{(6.67408) \times 10^{-11}) \times (5.97219 \times 10^{24})}{(6.3781366 \times 10^{6})^{2}} = -\,9.797999$$

Gravitational force against centrifugal force

$$-\frac{M_R m_s G}{R^2} = m_s \left(\frac{v^2}{R}\right)$$

$M_R =$ mass of primary within radius R
$m_s =$ mass of orbiting body
G = gravitational constant
v = velocity

Example: The International Space Station (ISS) has a mass of 419,725 kg and is in a nearly circular orbit at an altitude of 408,000 m above the surface of the Earth, whose radius is 6.38×10^{6} m. Since its centrifugal force must balance the gravitational force of the Earth, the velocity of the ISS is 7664 m/s.

$$-\frac{(5.97\times 10^{24})\times(419725)\times(6.67\times 10^{-11})}{[(6.38\times 10^{6})+408000]^2} = 419725\times\left(\frac{v^2}{6.38\times 10^{6})+408000}\right) \Longrightarrow v = 7664$$

Gravitational potential energy

$$\Omega = -\int_0^M \frac{G\,m(r)}{r} dm$$

G = gravitational constant
M = total mass
m(r) = mass as a function of distance from centre
r = distance from centre

Example: The density of the Sun and similar stars as a function of radius can be approximated by the expression:

$$\rho(r) = \rho_c\left(1-\frac{r}{R}\right)$$

where $\rho_c =$ the central density, r is the distance from the centre, and R is the radius. Making the substitution x = r/R allows the equation to be written as:

$\rho(r) = \rho_c\,(1-x)$. Then, *dr = Rdx.* The incremental mass is calculated as:

$$m(r) = 4\pi\int_0^r r_0^2\,\rho(r_0)\,dr_0 \;\; = 4\pi\int_0^r (R\,x_0)^2\,\rho(x_0)\,R\,dx_0$$

$$= 4\pi\,R^3\int_0^x x_0^2\,\rho_c(1-x_0)\;dx_0 \;\; = 4\pi\,R^3\,\rho_c\int_0^x x_0^2\,(1-x_0)\;dx_0$$

$$m(r) = 4\,\pi\,R^3\,\rho_c\,\left(-\frac{1}{4}\,x^4 + \frac{1}{3}\,x^3\right)$$

Total mass is found by setting *x = 1*:

$$M = 4\,\pi\,R^3\,\rho_c\,\left(-\frac{1}{4} + \frac{1}{3}\right) \Longrightarrow M = \frac{1}{3}\pi R^3\rho_c$$

$$\frac{m(r)}{M} = -3x^4 + 4x^3$$

$$m(r) = M(-3x^4 + 4x^3)$$

$$\frac{dm(r)}{dx} = M\frac{d}{dx}(-3x^4 + 4x^3)$$

$$dm(r) = M(-12x^3 + 12x^4)\,dx$$

Therefore, the gravitational potential energy is:

$$\Omega = -\int_0^M \frac{G\,m(r)}{r}dm$$

$$\Omega = -\int_0^1 \left(\frac{GM(-3x^4 + 4x^3)}{Rx})\right)(M(-12x^3 + 12x^2))dx$$

$\Omega = -\frac{26}{35}\frac{GM^2}{R}$. For the Sun, this equals $-2.8 \times 10^{41}\,J$.

Grey atmosphere temperature profile

$$T^4 = \frac{3}{4}T_e{}^4\left(\tau + \frac{2}{3}\right)$$

T = temperature
T_e = effective temperature (at $\tau = 2/3$)
τ = optical depth

Example: Light from a typical star comes from an **optical depth** of approximately 2/3. For the Sun, the effective temperature at this depth is 5777 K. Therefore, the temperature at the top of the atmosphere, where the optical depth is zero, is 4858 K.

$$T^4 = \frac{3}{4} \times (5777)^4 \times \left(\frac{2}{3}\right) \Longrightarrow T = 4858$$

Gyrofrequency (cyclotron frequency)

$$\omega_g = \frac{qB}{2\pi m}$$

m = particle mass
q = electron charge
B = magnetic field force

Example: The gyro frequency of a non-relativistic electron in interstellar space, where the magnetic field strength is approximately 3×10^{-10} T, is 8.4 Hz.

$$\omega_g = \frac{(1.6 \times 10^{-19}) \times (3 \times 10^{-10})}{2\pi \times (9.1 \times 10^{-31})} = 8.4\,Hz$$

Gyroradius (Larmor radius)

$$r_g = \frac{\gamma m c^2}{qcB} = \frac{E}{qcB} = \frac{m v_\perp c}{qB}$$

γ = Lorentz factor
m = particle mass
c = speed of light
q = electron charge
B = magnetic field force
E = particle energy
$v_\perp$ = particle velocity perpendicular to magnetic field

Example: Cosmic rays (which are, in fact, particles) having energies as high as 10^{20} eV (16 J) have been detected in interstellar space, where the magnetic field strength averages 3×10^{-10} T. The gyroradius of such a particle is approximately 10^{21} m, roughly the size of our Galaxy.

$$\frac{16}{(1.602 \times 10^{-19})(3 * 10^{8})(3 * 10^{-10})} = 1.1 \times 10^{21}$$

H

Hill radius

$$R_H = R\left(\frac{\rho}{\rho_{star}}\right)^{\frac{1}{3}}\left(\frac{a}{R_{star}}\right)$$

R = radius of the body in question
ρ = density of the body in question
ρ_{star} = density of the central star
a = distance of body in question from central star
R_{star} = radius of the central star

Example: The Hill radius of Jupiter is 7.67×10^{10} m. Any slowly moving object within that distance will be trapped in Jupiter's gravitational field.

$$6.9911 \times 10^7 \times \left(\frac{1326}{1408}\right)^{\frac{1}{3}} \times \left(\frac{7.785 \times 10^{11}}{6.957 \times 10^8}\right) = 7.67 \times 10^{10} \text{ m}$$

Heisenberg's Uncertainty Principle

$$\Delta x \Delta p \geq \frac{\hbar}{2}$$

Δx = uncertainty in position
Δp = uncertainty in momentum
$\hbar$ = Planck's constant divided by 2π

Example: If a 100 eV electron goes through a 10^{-6} m slit, the uncertainty in the angle of emergence (θ) is 1.95×10^{-5} rad.

$$p = \sqrt{2mE} = \sqrt{2 \times (9.1 \times 10^{-31}) \times 100\ \times (1.6 \times 10^{-19})} = 5.4 \times 10^{-24}\, kg\ m\ s^{-1}$$

$$\Delta p \approx \frac{\hbar}{\Delta x} = \frac{1.054 \times 10^{-34}}{1 \times 10^{-6}} = 1.054 \times 10^{-28}\, kg\ m\ s^{-1}$$

$$\Delta \theta \approx \frac{\Delta p}{p} = \frac{1054 \times 10^{-28}}{5.4 \times 10^{-24}} = 1.95 \times 10^{-5}$$

Hubble law

$$v = H_0 D$$

v = velocity
$H_0 =$ the Hubble constant
D = distance in megaparsecs

Example: One of the most distant Lyman-break galaxies, EGS-zs8-1, is receding at a velocity of 2.92×10^8 m/s. Taking the Hubble constant to be 73 km/s/Mpc, the distance of this galaxy at the time when the light now arriving at Earth was emitted was 4.0×10^6 Mpc or 13.04×10^9 light years.

$$\frac{2.92 \times 10^8}{73} = 4.0 \times 10^6$$

Hubble law, z factor

$$z = \frac{\lambda_o}{\lambda_e} - 1 = \sqrt{\frac{1 + \frac{v}{c}}{1 - \frac{v}{c}}} - 1$$

λ_o = observed wavelength
λ_e = emitted wavelength
v = velocity
c = speed of light

Example: One of the most distant Lyman-break galaxies, EGS-zs8-1, emits a Lyman-alpha line at a wavelength of 1215.67 $\mathring{A}$. The observed wavelength of this line from the galaxy is 10613.5 $\mathring{A}$, resulting in a z-factor of 7.73. This redshift indicates a recession velocity of 2.92×10^8 m/s, which is 97.4% of the speed of light.

$$z = \frac{10613.5}{1215.67} - 1 = 7.73 = \sqrt{\frac{1 + \frac{v}{2.998 \times 10^8}}{1 - \frac{v}{2.998 \times 10^8}}} - 1 \Longrightarrow v = 2.92 \times 10^8$$

Hydrostatic equilibrium

$$\frac{dP}{dr} = -\frac{GM(r)\rho(r)}{r^2}$$

P = pressure
r = distance from centre
G = gravitational constant
M(r) = mass within r
$\rho(r)$ = density within r

Example: A rough estimate of the Sun's central pressure can be made by noting that

$$\frac{dP}{dr} \sim \frac{P_{surface} - P_{core}}{R_{surface} - 0} \sim -\frac{P_{core}}{R_\odot}$$

Plugging this into the equation for hydrostatic equilibrium gives

$P_{core} = \dfrac{GM_r\rho}{R_\odot}$, which results in $2.7 \times 10^{14}\, Pa$. The actual value is $2.34 \times 10^{16}\, Pa$.

$$\frac{(6.67408 \times 10^{-11}) \times (1.989 \times 10^{30}) \times 1410}{6.9551 \times 10^{8}} = 2.7 \times 10^{14}$$

I

Ideal gas equation of state

$$PV = NRT$$

P = pressure
V = volume
N = number of moles
R = ideal gas constant (8.314 J/mol K)

Example: A useful form of this equation for many astrophysics calculations is:

$$P = \frac{\rho k T}{\mu m_H}$$

where P = pressure, $\rho =$ density, k = Boltzmann's constant, T = temperature, $\mu =$ mean molecular weight, and $m_H =$ mass of the hydrogen atom. For the Sun, the average density is $1410\, kg\ m^{-3}$, the central pressure is approximately $2.7 \times 10^{14}\, Pa$, and the mean molecular weight of the ionized gas is 0.62. The central temperature is then found to be $1.44 \times 10^{7}\, K$.

$$2.7 \times 10^{14} = \frac{1410 \times (1.38 \times 10^{-23}) \times T}{0.62 \times (1.67 \times 10^{-27})} \Longrightarrow T = 1.44 \times 10^{7}$$

Intensity: monochromatic (specific intensity, surface brightness, brightness)

$$I_\nu = \frac{F_\nu}{\Omega}$$

$F_\nu =$ monochromatic flux
$\Omega =$ solid angle on the sky

Example: The large radio-emitting galaxy M 87 occupies an approximately circular area on the sky with a diameter of about 7 minutes of arc. It has a measured monochromatic flux of 59,027 mJy. Its monochromatic intensity is, therefore, $1.8 \times 10^{-19}\, W\, m^{-2}\, Hz^{-1}\, sr^{-1}$.

$$\Omega = \frac{\pi}{4}\left(7 \times 60 \times \frac{1}{3600} \times \frac{\pi}{180}\right)^2 = 3.26 \times 10^{-6}\, sr$$

$$I_\nu = \frac{59.027 \times 10^{-26}}{3.26 \times 10^{-6}} = 1.8 \times 10^{-19}$$

Inverse Compton scattering vs. synchrotron radiation

$$\frac{L_{ic}}{L_s} = \left(\frac{T_{B_{max}}}{10^{12}}\right)^5 \left(\frac{\nu_{max}}{10^{8.5}}\right)$$

L_{ic} = inverse Compton luminosity
L_s = synchrotron luminosity
$T_{B_{max}}$ = maximum brightness temperature
ν_{max} = frequency at the peak of the spectrum

Example: The radio galaxy Cygnus A has a total flux density of $f_\nu = 10{,}950\, Jy$ at $\nu = 12.6$ MHz, the peak of its spectrum. This flux density is divided approximately equally between the two radio lobes, each of which has an angular diameter of $\theta_{FWHM} = 10'' = 4.85 \times 10^{-5}\, rad$. If synchrotron self-absorption is the only absorption process that is producing the spectral turnover, the ratio of L_{ic}/L_s for one radio lobe is 0.1, showing that radiation from inverse Compton scattering plays only a minor role compared with synchrotron radiation in this galaxy.

Calculate the intensity (1 jansky (Jy) = $10^{-26}\, W\, m^{-2}\, Hz^{-1}$):

$$I = \frac{10950 \times 10^{-26}}{\frac{\pi}{4}(4.85 \times 10^{-5})^2} = 5.93 \times 10^{-14}\, W\, m^{-2}$$

Test for **Rayleigh-Jeans limit**:

$$\frac{h \times (12.6 \times 10^6\, s^{-1})}{k} = 6.05 \times 10^{-4}\, K$$

(where h = Planck's constant, and k = Boltzmann's constant) This is well within the **Rayleigh-Jeans limit**, so the following equation for brightness temperature may be used:

$$T_B = \frac{c^2 \times (5.93 \times 10^{-14}\, J\, m^{-2})}{2 \times (12.6 \times 10^6\, s^{-1})^2 \times k} = 1.22 \times 10^{12}\, K$$

(where c = speed of light) Therefore,

$$\frac{L_{ic}}{L_s} = \left(\frac{1.22 \times 10^{12}}{10^{12}}\right)^5 \left(\frac{12.6 \times 10^6}{10^{8.5}}\right) = 0.1$$

Ionization fraction

$$f_i = \frac{\left(\frac{n_i}{n_{i-1}}\right)\left(\frac{n_{i-1}}{n_{i-2}}\right)\cdots\left(\frac{n_2}{n_1}\right)}{1 + \left(\frac{n_2}{n_1}\right) + \left(\frac{n_3}{n_2}\right)\left(\frac{n_2}{n_1}\right) + \left(\frac{n_4}{n_3}\right)\left(\frac{n_3}{n_2}\right)\left(\frac{n_2}{n_1}\right) + \cdots}$$

$n_i =$ ionization states

Example: For the ionization ratios $n_2/n_1 = 8.500$ and $n_3/n_2 = 0.0588$, the ionization fraction (n_1/n_{total}) is 0.1.

$$\frac{1}{1 + 8.5 + (0.0588)(8.5)} = 0.1000$$

K

Kepler's second law

$$\dot{A} = constant$$

$\dot{A}$ = rate of change of area swept out by the radius vector

Example: This law applies to any central-force-bound two-body system. One example is the Solar System.

The area of an infinitesimal triangle:

$$dA = \frac{1}{2}\vec{r} \times \vec{dr}$$

$$\dot{A} = \frac{dA}{dt} = \frac{1}{2}\vec{r} \times \vec{\dot{r}}$$

$$\ddot{A} = \frac{1}{2}\left(\vec{r} \times \vec{\ddot{r}} + \vec{\dot{r}} \times \vec{\dot{r}}\right) = 0$$

for any central force. Therefore, $\dot{A} = constant$.

Kepler's third law (Newton's version)

$$P^2 = \frac{4\pi a^3}{G\,(M+m)}$$

P = period
a = semi-major axis
G = gravitational constant
M, m = masses

Example: Io, a moon of Jupiter, orbits the planet in $1.53 \times 10^5\, s$. Its distance from Jupiter is $4.22 \times 10^8\, m$. With this information, and assuming that the mass of Io is insignificant compared with that of Jupiter, it is possible to calculate Jupiter's mass as $1.89 \times 10^{27}\, kg$.

$$\frac{4\pi^2}{6.67408 \times 10^{-11}} \times \frac{(4.22 \times 10^8)^3}{(1.53 \times 10^5)^2} = 1.89 \times 10^{27}$$

Kinetic energy of gas

$$KE = \frac{3}{2} NkT$$

N = number of particles
k = Boltzmann constant
T = temperature

Example: To calculate a rough estimate of the number of particles in the Sun, divide the mass of the Sun by the mass of the hydrogen atom. Then, since most of these atoms are ionized into a proton and an electron, multiply the result by 2. It follows that here are approximately 2.4×10^{57} particles in the Sun. The **virial temperature** of the Sun is $2.86 \times 10^6\, K$. Therefore, the kinetic energy of the solar gas is $1.4 \times 10^{41}\, J$.

$$\frac{3}{2}(2.4 \times 10^{57})(1.38 \times 10^{-23})(2.86 \times 10^6) = 1.4 \times 10^{41}$$

Kramers' opacity law

$$\kappa = (4.34 \times 10^{21}) \times Z(1 + X) \times \frac{\rho}{T^{3.5}}\, m^2\, kg^{-1}$$

Z = mass fraction of metals
X = mass fraction of hydrogen
ρ = density
T = temperature

Example: Kramers' opacity law is an empirical formula that gives an estimate of the gas opacity at temperatures over one million K. In the centre of the Sun, where the density is $1.62 \times 10^5\, kg\, m^{-3}$, the temperature is $1.6 \times 10^7\, K$, the hydrogen mass fraction is 0.7381, and the metals mass fraction is 0.0134, the Kramers' opacity is $1\, m^2\, kg^{-1}$.

$$\kappa = (4.34 \times 10^{21}) \times 0.0134(1 + 0.7381)\left(\frac{1.62 \times 10^5}{(1.6 \times 10^7)^{3.5}}\right) = 1$$

L

Landé g factor

$$g = 1 + \frac{j(j+1) + s(s+1) - l(l+1)}{2j(j+1)}$$

$j = |l \pm s|$ is the total angular momentum quantum number
s = spin quantum number
l = azimuthal quantum number

Example: The Landé g factor, from quantum mechanics, is used in a number of calculations, including **Zeeman splitting.** For example, the Fe I orbital $^5P_1$ (key: $n^{2s+1}L_j$) has quantum numbers j = 1, s = 2, and l = 1. Therefore, its Landé factor is 2.5.

$$1 + \frac{1(1+1) + 2(2+1) - 1(1+1)}{2 \times 1 \times (1+1)} = \frac{5}{2}$$

Larmor's formula for power

$$P = \frac{q^2\ddot{x}^2}{6\pi\varepsilon_0 c^3} = \frac{\mu_0 q^2 \ddot{x}^2}{6\pi c}$$

q = electron charge
$\ddot{x}$ = acceleration
ε_0 = permittivity of free space
c = speed of light
μ_0 = permeability of free space

Example: If an electron travelling at a speed of $10^8\, m\, s^{-1}$ strikes an atom and comes to rest after travelling a distance of $3 \times 10^{-9} m$, the fraction of the energy radiated away is 2.1×10^{-10}.

Calculate the time required for the electron to decelerate:

$$t = \frac{\dot{x}}{\ddot{x}}$$

The total energy radiated during this period is

$$E = Pt = \frac{\mu_0 q^2 \ddot{x}^2}{6\pi c}\frac{\dot{x}}{\ddot{x}} = \frac{\mu_0 q^2 \ddot{x}\dot{x}}{6\pi c}$$

The fraction of the electron's kinetic energy radiated away:

$f = \dfrac{2E}{m\dot{x}^2} = \dfrac{\mu_0 q^2 \ddot{x}}{3\pi m c\dot{x}}$, where m is the mass of the electron.

Find the acceleration of the electron:

$$\dot{x} = \frac{1}{2}\ddot{x}t^2 \rightarrow \ddot{x} = \frac{\dot{x}^2}{2x} = 1.67 \times 10^{18}\, ms^{-2}\text{s}$$

Putting these values into the equation for *f*, above:

$$\frac{(1.602 \times 10^{-19})^2 \times (4 \times \pi \times 10^{-7}) \times (1.67 \times 10^{18})}{3 \times \pi \times (9.109 \times 10^{-31}) \times (3 \times 10^8) \times 10^5} = 2.1 \times 10^{-10}$$

Hence, a very small amount of energy is lost to radiation as a result of electron collisions.

Light pressure: total reflection $P_{refl} = \dfrac{2f}{c}\cos^2\theta \rightarrow P = \dfrac{F}{A} \rightarrow F = \dfrac{A2L}{c4\pi r^2}$

f = luminous flux
c = speed of light
θ = angle of incidence
F = force
A = area of 100% reflective object
L = luminosity ($= 4\pi r^2 f$)
r = distance from light source to reflecting object

Example: A 100% reflective **solar sail** with a radius of 600 m is located in the Earth's orbit. The sail points directly toward the Sun. The light pressure on this sail is 10.3 N.

$$\frac{\pi \times (600)^2 \times 2 \times (3.845 \times 10^{26})}{(3 \times 10^8) \times 4 \times \pi \times (1.496 \times 10^{11})^2} = 10.3$$

(Note: The light pressure on a totally absorbent object is half this amount: $P_{abs} = \frac{f}{c}\cos^2\theta$).

Line depth

$$A_\lambda = 1 - \frac{F_\lambda}{F_c}$$

F_λ = flux of the line
F_c = flux of the continuum

Example: The line depth of a rectangular line with a line flux that is 2/3 the continuum flux is 1/3. This information can be used to calculate the **equivalent width** of the line.

$$1 - \frac{\frac{2}{3}F_c}{F_c} = \frac{1}{3}$$

Local sidereal time

$$LST = GST + \lambda = HA + RA$$

GST = Greenwich Sidereal Time
λ = local longitude
HA = hour angle
RA = right ascension

Example: At Universal Time (UT) 4h 32m 38s on 29 March 2019, the sidereal time at Greenwich was 17h 02m 03s. In Vancouver (longitude = -123.08), the local sidereal time was 8h 53m 43s.

$$\begin{aligned} LST &= 17h\ 02m\ 03s + (-122.08^\circ) \\ &= 255.5125^\circ - 122.08^\circ = 133.4325^\circ \\ &= 8h\ 53m\ 43s \end{aligned}$$

Lorentz factor

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

v = velocity
c = speed of light

Example: Muons are produced in the upper atmosphere by cosmic rays striking atmospheric atoms. Muons have a half-life of $1.56 \times 10^{-6}\, s$. If they are formed at an altitude of $10^4\, m$ and descend vertically at a speed of 98% the speed of light, only 0.3 out of a million would be expected to reach the surface of the Earth. In fact, 49,000 out of a million reach the surface. This is a confirmation of special relativity.

Non-relativistic time of descent:

$$t = \frac{10^4}{(0.98)(3 \times 10^8)} = 34 \times 10^{-6}\, s = 21.8 \text{ half-lives}$$

$\frac{I}{I_0} = 2^{-21.8} = 0.27 \times 10^{-6}$, or about 0.3 out of a million.

Relativistic time of descent:

At the speed of 0.98 c, the Lorentz factor is

$$\gamma = \frac{1}{\sqrt{1 - \frac{(0.98c)^2}{c^2}}} = 5$$

The muons' half-life is dilated to $t = \gamma\, t_0 = 7.8\, \mu s$ or 4.36 half-lives:

$\frac{I}{I_0} = 2^{-4.36} = 0.049$, or about 49,000 out of a million, which is what is observed.

Lorentzian profile

$$\phi = \frac{\frac{\Gamma}{4\pi^2}}{(\nu - \nu_0)^2 - \left(\frac{\Gamma}{4\pi}\right)^2}$$

Γ = radiative damping constant
ν_0 = natural frequency of a transition
ν = neighbouring frequency

Example: FWHM for a Lorentzian profile is equal to $\frac{\Gamma}{2\pi}$.

At peak value, $\nu = \nu_0$. Therefore, $\phi = \dfrac{4}{\Gamma}$. At half peak value, $\phi = \dfrac{2}{\Gamma}$.

Inserting this value into the original equation and solving for v gives the frequency at half width at half maximum:

$$\frac{2}{\Gamma} = \frac{\frac{\Gamma}{4\pi^2}}{(\nu - \nu_0)^2 - \left(\frac{\Gamma}{4\pi}\right)^2} \Longrightarrow \nu = \nu_0 \pm \frac{\Gamma}{4\pi}$$

Therefore the full width at half maximum is twice that value, centred on ν_0: $\dfrac{\Gamma}{2\pi}$.

Luminosity distance

$$d_L = (1+z)\frac{c}{H_0}\int_0^z \frac{dz}{[\Omega_M(1+z)^3 + \Omega_L]^{1/2}}$$

$$m - M = 5\log_{10}\left(\frac{d_L}{1\,Mpc}\right) + 25$$

z = Hubble redshift factor
c = speed of light
H_0 = Hubble constant
Ω_M = the mean present day fractional energy density of all forms of matter
Ω_L = the ratio between the energy density due to the cosmological constant and the critical density of the universe
m = apparent magnitude
M = absolute magnitude

Example: These equations apply to objects located at significant z-distances. In the case of a galaxy at z = 0.8 and with an absolute magnitude in the B filter of -19.6, and assuming the Universe is described by $\Omega_M = 0.3$, $\Omega_\lambda = 0.7$, and $H_0 = 70\,km\,s^{-1}\,Mpc^{-1}$, the galaxy's expected apparent magnitude, neglecting *k* corrections, band-pass corrections, and dust effects, would be 23.4 in the B filter, and its luminosity distance is $3.94 \times 10^9\,pc$ (Note: These equations assume bolometric magnitudes. Otherwise, further corrections are needed.)

$$d_L = (1+0.8)\frac{(3\times 10^8)}{70000}\int_0^{0.8}\frac{dz}{[0.3(1+0.8)^3+0.7]^{1/2}} = 3.94\times 10^9\, pc$$

$$m = 5\log_{10}\left(\frac{(3.94\times 10^9)}{10^6}\right) + 25 + (-19.6) = 23.4$$

Luminous flux ratio and brightness ratio

$$R_B = 100^{\frac{M_1 - M_2}{5}}$$

$$R_F = R_B \times \left(\frac{\Omega_1}{\Omega_2}\right)$$

$R_B =$ brightness ratio
$M_1, M_2 =$ absolute magnitudes of two objects
$R_F =$ luminous flux ratio
$\Omega_1, \Omega_2 =$ solid angles subtended by two objects

Example: The galaxies M31 and M83 have absolute magnitudes, respectively, of -21.5 and -20.31. M31 subtends a solid angle on the sky of 2280.18 square arcminutes, while M83 subtends a solid angle of 29.08 square arcminutes. See **"Solid angle subtended by an ellipse"**.) As a result, M31 is only about three times as bright as M83 but has about 26 times the flux density.

$$R_B = 100^{\frac{-20.31-(-21.5)}{5}} = 2.99$$

$$R_F = 2.99\times\left(\frac{29.08}{2280.18}\right) = 0.038 \Longrightarrow \frac{1}{0.038} = 26.3$$

M

M-sigma relation for black hole mass

$$M_{BH} = 1.9\left(\frac{\sigma}{200\,km\ s^{-1}}\right)^{5.1}\ 10^8\,M_\odot$$

σ = velocity dispersion in galactic bulge
$M_\odot$ = solar mass

Example: The velocity dispersion in the bulge of the galaxy NGC 4258 is 140 km/s. Therefore, the mass of the black hole at the centre of the galaxy is 3.8×10^7 times the mass of the Sun. (This is an empirical formula.)

$$M_{BH} = 1.9\left(\frac{140}{200}\right)^{5.1}\ 10^8\,M_\odot = 3.8 \times 10^7\,M_\odot$$

Magnitude to flux

$$m_1 - m_2 = -\,2.5\ \log\left(\frac{f_1}{f_2}\right)$$

$$\left(\frac{f_1}{f_2}\right) = 10^{-0.4(m1-m2)}$$

m = magnitudes
f = fluxes

Example: If one star is 5 magnitudes brighter than another, its flux is 100 times greater.

$$10^{-0.4(0-5)} = 100$$

Mass of object orbiting another $m_1 = \frac{m_2 \, r_2}{r_1}$

m = mass
r = distance to barycentre

Example: Pluto's moon Charon orbits Pluto (mass: $1.309 \times 10^{22} \, kg$) at a distance of $1.96 \times 10^7 \, m$. The barycentre is $2.11 \times 10^6 \, m$ from the centre of Pluto. Therefore, the mass of Charon is $1.58 \times 10^{21} \, kg$.

$$\frac{(1.309 \times 10^{22}) \times (2.11 \times 10^6)}{[(1.96 \times 10^7) - (2.11 \times 10^6)]} = 1.58 \times 10^{21}$$

Mass-luminosity relation

$$\frac{L}{L_\odot} = 0.23 \left(\frac{M}{M_\odot} \right)^{2.3} \quad (M < 0.43 M_\odot)$$

$$\frac{L}{L_\odot} = \left(\frac{M}{M_\odot} \right)^{4} \quad (0.43 M_\odot < M \leq 2 M_\odot)$$

$$\frac{L}{L_\odot} = 1.5 \left(\frac{M}{M_\odot} \right)^{3.5} \quad (2 M_\odot < M \leq 20 M_\odot)$$

$$\frac{L}{L_\odot} = 32000 \left(\frac{M}{M_\odot} \right) \quad (M > 55 M_\odot)$$

L = **luminosity**
M = mass of star
$M_\odot =$ solar mass

*Example: (*These equations are useful mostly for indicating the mass ranges and luminosity ranges of stars of the different types on the Hertzsprung-Russell diagram. They apply only to Main Sequence stars. For individual stars, the equations give a very approximate value at the most.) The star Rigel is thought to be 47,000 times as luminous as the Sun. It is expected, therefore, to have 19 times the Sun's mass. The accepted value is 17.

$$47000 = 1.5 \times M^{3.5} \implies M = 19$$

Mass-radius relation, neutron star

$$R_{ns} \approx \frac{(18\pi)^{2/3}}{10} \frac{(\hbar)^2}{GM_{ns}^{1/3}} \left(\frac{1}{m_H}\right)^{8/3}$$

R_{ns} = radius of neutron star
$\hbar$ = reduced Planck constant
G = gravitational constant
M_{ns} = mass of neutron star

Example: A typical neutron star has a mass of about 1.4 times the mass of the Sun. Its expected radius is, therefore, 4445 m. The commonly accepted values vary from about 5000 m to 6000 m.

$$R_{ns} \approx \frac{(18\pi)^{2/3}}{10} \frac{(1.0547 \times 10^{-34})^2}{(6.67408 \times 10^{-11})(1.4 \times 1.989 \times 10^{30})^{1/3}} \left(\frac{1}{(1.67 \times 10^{-27})}\right)^{8/3} = 4445$$

Mass-radius relation: Main Sequence stars

$$R = M^{0.738}$$

R = radius
M = mass

Example: The mass of Sirius is about 2.24 times that of the Sun. Its expected radius is, therefore, 1.8 times that of the Sun. The accepted value is 1.711.

$(2.24)^{0.738} = 1.8$

Mass-radius relation, white dwarf

$$R = \frac{3}{2}\left(\frac{6\pi^2}{g^2}\right)^{1/3} \frac{\hbar^2}{Gm_e(\mu m_p)^{5/3}} M^{-1/3}$$

g = degeneracy factor for spin (≈ 2)
$\hbar$ = reduced Planck constant
m_e = mass of electron
μ = reduced mass
m_p = mass of proton

Example: The white dwarf Sirius B has about 98% the mass of the Sun. The reduced mass is approximately 2. Therefore, its radius is approximately 4.9×10^6 m, or about 3/4 the size of the Earth. The currently accepted figure is $5.84 \times 10^6\ m$.

$$R = \frac{3}{2}\left(\frac{6\pi^2}{2^2}\right)^{1/3} \frac{(1.05457 \times 10^{-34})^2}{(6.674 \times 10^{-11})(9.109 \times 10^{-31})(2 \times 1.67 \times 10^{-27})^{5/3}} (0.98 \times 1.989 \times 10^{30})^{-1/3} = 4.9 \times 10^6$$

$$log_{10}\left(\frac{M_{WD}}{M_{Sun}}\right) = 8.04 + 3.84 \times log_{10}\left(\frac{85}{200}\right) \Longrightarrow M_{WD} = 4.1 \times 10^6$$

Maxwell-Boltzmann velocity distribution function

$$n_v dv = n\left(\frac{m}{2\pi kT}\right)^{3/2} e^{-mv^2/2kT} 4\pi v^2 dv$$

n_v = number of particles moving at velocity *v*
m = mass of particle
k = Boltzmann's constant
T = temperature

Example: This formula gives the number of gas particles per unit volume having speeds between v and v+dv, and displays as a probability distribution. The fraction of particles with speeds between v_1 and v_2 is equal to the area under the curve between the two speeds. Thus, if a hydrogen gas is at a temperature of 10000 K, to find the number of atoms with speeds between $v_1 = 2 \times 10^4\, m\, s^{-1}$ and $v_2 = 2.5 \times 10^4\, m\, s^{-1}$, it is necessary to integrate the function between these two limits. It turns out that about 12.8% of the atoms fall within this range.

$$\frac{N}{N_{Total}} = \frac{1}{n}\int_{v_1}^{v_2} n_v dv$$

$$= \left(\frac{m}{2\pi kT}\right)^{3/2} \int_{v1}^{v2} e^{-mv^2/2kT} 4\pi v^2 dv$$

$$= \left(\frac{1.6737236 \times 10^{-27}}{2\pi(1.380648510 \times 10^{-23})10000}\right)^{3/2} \int_{2\times10^4}^{2.5\times10^4} e^{-(1.6737236\times10^{-27})v^2/2(1.380648510\times10^{-23})10000)} 4\pi v^2 dv$$

$$= 0.1276$$

Mean free path

$$l = \frac{1}{n\sigma}$$

$n = \dfrac{\rho}{m_H}$ = number density of hydrogen
ρ = density
m_H = mass of hydrogen atom
σ = collisional cross-section

Example: In the photosphere of the Sun, which is primarily hydrogen, the density is about $2.1 \times 10^{-4}\, kg\, m^{-3}$. Therefore, the number density of hydrogen there is

$$n = \frac{2.1 \times 10^{-4}}{1.67 \times 10^{-27}} = 1.26 \times 10^{23}\, m^{-3}$$

It follows that the mean free path of a hydrogen atom in the solar photosphere is

$$\frac{1}{(1.26 \times 10^{23}\, m^{-3})(3.52 \times 10^{-20}\, m^2)} = 2.25 \times 10^{-4}\, m$$

Mean molecular weight

$$\mu = \frac{\langle m \rangle}{m_A} = \frac{1}{m_A N} \sum m_j$$

For proportions of X, Y, and Z:

Neutral gas:

$$\frac{1}{\mu_n} \simeq X + \frac{1}{4}Y + \left\langle \frac{1}{A} \right\rangle_n Z$$

Fully ionized gas:

$$\frac{1}{\mu_i} \simeq 2X + \frac{3}{4}Y + \left\langle \frac{1+z}{A} \right\rangle_i Z$$

$\langle m \rangle =$ mean mass of free particles
$m_A =$ mass of one atom of the target species.
$N =$ total number of all free particles
$m_j =$ mass of the jth particle
$X =$ proportion of hydrogen
$Y =$ proportion of helium
$Z =$ proportion of metals
$A_j = m_j / m_H$
$m_H =$ mass of hydrogen atom
$1 + z_j =$ nucleus plus number of free electrons resulting from ionization

Example: The mean molecular weight of a pure hydrogen gas that is completely ionized (equal number of discrete electrons and protons) is 1/2.

$$\mu = \frac{1}{m_H N}\left[\frac{N}{2}m_e + \frac{N}{2}m_p\right]$$

$$= \frac{1}{1.6727 \times 10^{-27} N}\left[\frac{N}{2}9.109 \times 10^{-31} + \frac{N}{2}1.67 \times 10^{-27}\right] = 0.5$$

Mean molecular weight: star

For fully ionized gas:

$$\mu = \left[2X + \frac{3}{4}Y + \frac{1}{2}Z\right]^{-1}$$

For neutral gas:

$$\mu = \left[X + \frac{Y}{4} + \left\langle \frac{1}{A_j} \right\rangle Z\right]^{-1}$$

$X =$ mass fraction of hydrogen
$Y =$ mass fraction of helium
$Z =$ mass fraction of metals
$\left\langle \frac{1}{A_j} \right\rangle =$ average over metals

Example: The solar values are as follows: X = 0.7; Y = 0.28; Z = 0.02; $A_j = 15.5$. Therefore, the mean molecular weight of fully ionized gas in the Sun is 0.62, and that of neutral gas is 1.3.

$$\mu = \left[2(0.7) + \frac{3}{4}(0.28) + \frac{1}{2}(0.02)\right]^{-1} = 0.62 \text{ (fully ionized gas)}$$

$$\mu = \left[0.7 + \frac{0.28}{4} + \left\langle \frac{1}{15.5} \right\rangle (0.02)\right]^{-1} = 1.3 \text{ (neutral gas)}$$

Metallicity

$$M = \log_{10}\left(\frac{N_{Fe}}{N_H}\right)_{star} - \log_{10}\left(\frac{N_{Fe}}{N_H}\right)_{Sun}$$

N_{Fe} = number of iron atoms
N_H = number of hydrogen atoms

Example: The red-giant star Arcturus has a $\left(\frac{N_{Fe}}{N_H}\right)$ value of 0.004. That of the Sun is measured to be 0.0196. Therefore, the metallicity of Arcturus is -0.69, or about 20% that of the Sun.

$$M = \log_{10}(0.004) - \log_{10}(0.0196) = -0.69$$

Minimum density: uniform density sphere

$$\rho > \frac{2\pi}{GP^2}$$

G = gravitational constant
P = period

Example: One of the fastest-spinning pulsars has a rotation period of $1.4 \times 10^{-3}\, s$. In order to keep from flying apart, it must have a density greater than $4.8 \times 10^{16}\, kg\, m^{-3}$. This is approximately the density of an atomic nucleus.

$$\rho > \frac{2\pi}{(6.67408 \times 10^{-11}) \times (1.4 \times 10^{-3})^2} = 4.8 \times 10^{16}$$

Moment of inertia: sphere

$$I = \frac{2}{5}MR^2$$

$$I = \frac{L}{\omega}$$

M = mass

R = radius
L = **angular momentum**
ω = angular frequency

Example: The Crab pulsar has a mass of $2.784 \times 10^{30}\,kg$ and a radius of $10^4\,m$. Therefore, its moment of inertia is $1.1 \times 10^{38}\,kg\,m^2$.

$$I = \frac{2}{5}(2.784 \times 10^{30})(10^4)^2 = 1.1 \times 10^{38}$$

Momentum of a particle

$$p = \gamma m v$$

γ = the Lorentz factor
m = mass
v = velocity

Example: An electron moving at 0.985 times the speed of light has a momentum of $1.56 \times 10^{-21}\,kg\,m\,s^{-1}$.

$$\frac{1}{\sqrt{1-\left(\frac{0.985c}{c}\right)^2}} \times (9.11 \times 10^{-31}\,kg) \times 0.985 \times (3 \times 10^8\,m\,s^{-1}) = 1.56 \times 10^{-21}\,kg\,m\,s^{-1}$$

Momentum of a photon

$$p = \frac{E}{c} = \frac{h\nu}{c} = \frac{h}{\lambda}$$

$E =$ **energy of electromagnetic radiation**
$c =$ speed of light
$h =$ Planck's constant
$\nu =$ frequency
$\lambda =$ wavelength

Example: A photon with a frequency of $3 \times 10^{14}\,s^{-1}$ (infrared) has a momentum of $6.6307 \times 10^{-28}\,kg\,m\,s^{-1}$.

$$\frac{(6.626 \times 10^{-34})(3 \times 10^{14})}{(3 \times 10^8)} = 6.6307 \times 10^{-28}$$

N

Neutron star luminosity

$$L = \left(\frac{8}{5}\right) \pi^2 M R^2 P^{-3} \left(\frac{dP}{dt}\right)$$

M = mass
R = radius
P = period
$\frac{dP}{dt}$ = spin-down rate

Example: The Crab pulsar has a mass of 1.5 times the mass of the Sun, a period of 33 ms, a radius of 10 km, and a spin-down rate of $4.2 \times 10^{-13}\, s/s$. Its predicted luminosity is, therefore, $5.5 \times 10^{31}\, W$. The observed luminosity is $2 \times 10^{31}\, W$, which means that not all of the spin-down energy is converted to luminosity.

$$L = \left(\frac{8}{5}\right) \pi^2 (1.5 \times 1.99 \times 10^{30})(10000)^2 \left(\frac{33}{1000}\right)^{-3} (4.2 \times 10^{-13}) = 5.5 \times 10^{31}$$

Newton's second law

$$F = m \times a$$

F = force
m = mass
a = acceleration

Example: If an electron accelerates constantly from $3 \times 10^5\, m\, s^{-1}$ to $7 \times 10^5\, m\, s^{-1}$ within a distance of 5 cm, the force required to produce this acceleration is $3.64 \times 10^{-18}\, N$.

Time during which the electron accelerates (distance divided by average velocity):

$$t = \frac{0.05\,m}{5 \times 10^5\,m\,s^{-1}} = 10^{-7}\,s$$

Acceleration (change in velocity divided by the time):

$$a = \frac{4 \times 10^5\,m\,s^{-1}}{10^{-7}\,s} = 4 \times 10^{12}\,m\,s^{-2}$$

$$F = (9.11 \times 10^{-31}\,kg) \times (4 \times 10^{12}\,m\,s^{-2}) = 3.64 \times 10^{-18}\,N$$

Number of photon interactions in a gas $N = \tau^2$

τ = optical depth

Example: The optical depth of the whole Sun is estimated to be 9.8×10^{10}. A photon moving from the centre of the Sun to the surface would experience about 9.6×10^{21} interactions.

$$(9.8 \times 10^{10})^2 = 9.6 \times 10^{21}$$

O

Oort formula for Galactic rotation

$$\left.\frac{d\Theta}{dR}\right|_{R_0} = -(A+B)$$

$$A = -\frac{1}{2}\left[\left.\frac{d\Theta}{dR}\right|_{R_0} - \frac{\Theta_0}{R_0}\right]$$

$$A = -\frac{1}{2}\left[\left.\frac{d\Theta}{dR}\right|_{R_0} + \frac{\Theta_0}{R_0}\right]$$

$\frac{d\Theta}{dR}$ = rate of Galactic rotation at distance *R* from centre

R_0 = distance of Sun from Galactic centre

Θ_0 = rate of Sun's rotation around the Galactic centre

Example: The currently accepted figures for *A* and *B* are $15.3 \pm 0.4\, km\ s^{-1}\, kpc^{-1}$ and $-11.9 \pm 0.4\, km\ s^{-1}\, kpc^{-1}$, respectively. This leads to a Galactic rotation rate at the Sun's distance from the Galactic centre of $\left.\frac{d\Theta}{dR}\right|_{R_0} = -3.4\, km\ s^{-1}\, kpc^{-1}$, implying that the rotation rate at the Sun's distance from the galactic centre is decreasing with distance.

$$\left.\frac{d\Theta}{dR}\right|_{R_0} = -\,(15.3 - 11.9) = -\,3.4$$

P

Partition function

$$Z = \sum_{n}^{n_{max}} g_n \, e^{-\left(\frac{\Delta E}{kT}\right)}$$

n = atomic energy levels
g = statistical weight ($2n^2$ for hydrogen)
ΔE = difference in energy between two levels
k = Boltzmann's constant
T = temperature

Example: The partition function of the hydrogen atom at the solar surface temperature of 5777 K is almost exactly 2. This can be seen by calculating the summation out to n = 3. The first term is equal to the statistical weight for n = 1, since the exponential term is equal to 1. Subsequent terms are close to zero on account of vanishingly small exponentials. The first three energy levels of the hydrogen atom are -13.6 eV, -3.4 eV, and -1.5 eV, respectively. Therefore, the energy differences are: n1 to n1, 0; n1 to n2, 10.2; and n1 to n3, 12.1.

$$2e^0 + 8e^{-\left(\frac{10.2 \times (1.602 \times 10^{-19})}{(1.3806 \times 10^{-23}) \times 5777}\right)} + 18e^{-\left(\frac{12.1 \times (1.602 \times 10^{-19})}{(1.3806 \times 10^{-23}) \times 5777}\right)} = 2.00000001$$

Photon frequency from energy-level change

$$\nu = \frac{E_i - E_f}{h}$$

E_i = initial energy
E_f = final energy
h = Planck's constant

Example: (This formula is an extension of the formula for the **energy of electromagnetic radiation**.) In a hydrogen atom, if the electron jumps from the n = 2 energy level (-3.4 eV) to the n = 1 energy level (-13.6 eV), it releases a photon with a frequency of $2.47 \times 10^{15} Hz$.

$$\nu = \frac{-3.4 - (-13.6)}{6.62607 \times 10^{-34}} = 2.47 \times 10^{-7}$$

Poynting flux (Poynting vector)

$$\left| \vec{S} \right| = \frac{\left| \vec{E} \times \vec{B} \right|}{\mu_0} = \frac{E_0 \, B_0}{\mu_0} \cos^2(\vec{k} \cdot \vec{r} - \omega \, t)$$

E = electric field
B = magnetic field
μ_0 = permeability of free space
k = wave number
r = distance
ω = angular frequency
t = time

Example: The time-averaged intensity (using the fact that $\langle \cos^2 \rangle = 1/2$) is $\frac{c}{2} \left(\frac{B_0^2}{2\,\mu_0} + \frac{\varepsilon_0 \, E_0^2}{2} \right)$.

$$\langle S \rangle = \frac{E_0 \, B_0}{2\,\mu_0} = \frac{c \, E_0}{2\,\mu_0 \, c^2} = \frac{1}{2} \, c \, \varepsilon_0 \, E_0^2 = \frac{1}{2} \, c \, \frac{B_0^2}{\mu_0} = \frac{c}{2} \left(\frac{B_0^2}{2\,\mu_0} + \frac{\varepsilon_0 \, E_0^2}{2} \right)$$

Poynting-Robertson effect

$$t = \frac{4\pi\rho c^2}{3\,L_\odot} \, R \, r^2$$

t = time for orbiting dust grain to spiral into the Sun
ρ = density of dust grain
c = speed of light
$L_\odot$ = solar luminosity
R = radius of dust grain
r = initial distance of dust grain from Sun

Example: A dust grain of radius $10^{-4}\,m$ and density of $10^3\,kg\,m^{-3}$ would take approximately 70,000 years to spiral from a distance of 1 AU into the Sun.

$$\frac{4 \times \pi \times 10^3 \times (3 \times 10^8)^2}{3 \times (3.828 \times 10^{26})} \times 10^{-4} \times (1.496 \times 10^{11})^2 = 2.2 \times 10^{12}\,s \approx 70{,}000\,years$$

Pressure scale height

$$H_p = \frac{P}{\rho\, g_{av}}$$

where $g_{av} = \dfrac{G\left(\frac{M}{2}\right)}{\left(\frac{r}{2}\right)^2}$

P = pressure
ρ = density
g_{av} = average gravitational acceleration
G = gravitational constant
M = mass of star
r = radius of star

Example: The pressure scale height of the whole Sun is calculated to be $7.2 \times 10^8\, m$. (A more detailed calculation give $5 \times 10^8\, m$ as the commonly accepted value.)

$$g_{av} = \frac{6.673 \times 10^{-11} \left(\frac{1.99 \times 10^{30}}{2}\right)}{\left(\frac{6.99 \times 10^8}{2}\right)^2} = 543.6\, m\, s^{-2}$$

$$H_p = \frac{1.1 \times 10^{15}}{1410 \times 544} = 7.2 \times 10^8\, m$$

Pulsar magnetic field strength at pole

$$B_{pole} = \frac{1}{2\pi R^3 \sin(\theta)} \left(\frac{3\,\mu_0\,(c)^3\, I\, P\, \dot{P}}{2\pi}\right)^{\frac{1}{2}}$$

R = radius
θ = angle between magnetic pole and rotation axis
μ_0 = permeability of free space
c = speed of light
I = moment of inertia
P = period
$\dot{P}$ = spin-down rate

Example: For the Vela pulsar (PSR J0835-4510), the period is 0.08933 s, and the spin-down rate is 1.25008×10^{-13}. Theta equals $\frac{3}{30}\pi\, rad$; $\mu_0 = \frac{\pi}{2.5 \times 10^6}\, N A^{-2}$; and its moment of inertia is $1.11384 \times 10^{38}\, kg\, m^2$. Therefore, its magnetic field strength at the poles is $7.27 \times 10^8\, T$.

$$\frac{1}{2(10^4)^3\left(\sin\frac{13}{30}\pi\right)}\left(\frac{3\left(\frac{\pi}{2.5\times10^6}\right)(2.99\times10^8)^3\,(1.11384\times10^{38})\,0.08932\,(1.25008\times10^{-13})}{2\pi}\right)^{\frac{1}{2}} = 7.27\times10^8$$

Pulsar magnetic field strength at surface

$$B_{surf} = 3.2\times10^{15}\,(P\,\dot{P})^{\frac{1}{2}}$$

P = period
$\dot{P}$ = spin-down rate

Example: For the Vela pulsar (PSR J0835-4510), the period is 0.08933 s, and the spin-down rate is 1.25008×10^{-13}. Therefore, the magnetic field strength at the surface is $3.38\times10^8\,T$.

$$3.2\times10^{15}\,[0.08933\times(1.25008\times10^{-13})]^{\frac{1}{2}} = 3.38\times10^8$$

Q

Quantised energy levels: hydrogen

$$E_n = -R_\infty \left(\frac{1}{n^2}\right)$$

R_∞ = Rydberg unit of energy for hydrogen (= 13.605693 eV)
n = energy level (eV)

Example: The energy of the third level of the hydrogen atom is -1.5 eV.

$$-13.605693 \left(\frac{1}{3^2}\right) = -1.5$$

R

Radiation force of the Sun

$$F_R = \frac{\pi\, \sigma\, r^2\, R_\odot^2\, T_\odot^4}{c\, d^2}$$

σ = Stefan-Boltzmann constant
r = radius of target object
$R_\odot$ = radius of the Sun
$T_\odot$ = temperature of solar surface
c = speed of light
d = distance of target from Sun

Example: The force of solar radiation on the whole Earth is $5.8 \times 10^8\, N$.

$$\frac{\pi\, (5.67 \times 10^{-8})\, (6.38 \times 10^6)^2\, (6.96 \times 10^8)^2\, (5777)^4}{(3 \times 10^8)\, (1.496 \times 10^{11})^2} = 5.8 \times 10^8$$

Radiative flux (Stefan-Boltzmann law; integrated flux)

$$F = \frac{L}{4\, \pi\, r^2} = \sigma\, T^4$$

L = **luminosity**
r = distance from source
σ = Stefan-Boltzmann constant
T = temperature

Example: The luminosity of the Sun is $3.846 \times 10^{26}\, W$. Therefore, the radiative flux at Earth is $1368\, W\, m^{-2}$.

$$\frac{3.846 \times 10^{26}}{4\,\pi\,(1.496 \times 10^{11})^2} = 1368\,W\,m^{-2}$$

Radius of orbit of object orbiting star

$$r_p = \left(\frac{P^2}{\left(\frac{4\,\pi^2}{G\,M_s}\right)}\right)^{\frac{1}{3}}$$

Example: The first exoplanet that was discovered orbits the star 51 Pegasi with a period of $3.65472 \times 10^5\,s$. The mass of the star is $2.10834 \times 10^{30}\,kg$. Therefore, the radius of its orbit is $7.8 \times 10^9\,m$.

$$\left(\frac{(3.65472 \times 10^5)^2}{\left(\frac{4\,\pi^2}{(6.67 \times 10^{-11})\,(2.10834 \times 10^{30}}\right)}\right)^{\frac{1}{3}} = 7.8 \times 10^9$$

Radius of star's orbit about barycentre

$$r_s = \frac{P\,v_s}{2\,\pi}$$

P = orbital period
v_s = orbital velocity of star

Example: The star 51 Pegasi is observed to "wobble" around a centre-of-mass point with a velocity of 54.87 m/s and a period of $3.65472 \times 10^5\,s$, thus indicating the presence of an orbiting object of significant mass. The distance between the centre of the star and the centre of mass is, therefore, $3.19 \times 10^6\,m$, or about 13% of the distance between Mercury and the Sun.

$$\frac{(3.65472 \times 10^5)\,(54.87)}{2\,\pi} = 3.19 \times 10^6$$

Rayleigh-Jeans approximation ($h\nu/kT < 0.19$**)**

$$B_\lambda(T) = \frac{2ckT}{\lambda^4} = \frac{2\nu^2 kT}{c^2} = \frac{2kT}{\lambda^2}$$

c = speed of light
k = Boltzmann constant
T = temperature
λ = wavelength
ν = frequency

Example: (This formula is an approximation of **Planck's law** for the long-wavelength side of the spectrum and is especially useful for radio observations.) The radio source the Orion A molecular cloud has an intensity ($B_\lambda = I_\lambda$) of 400 Jansky at a wavelength of 1.3 cm. It occupies a solid angle of $5.28 \times 10^{-7}\,sr$ on the sky. Total flux equals intensity times solid angle:

$$F = B_\lambda\,\Omega$$

Therefore, the brightness temperature of the Orion A molecular cloud at this wavelength is 46 K.

$$T = \frac{B_\lambda\,\lambda^2}{2k\,\Omega} = \frac{(4 \times 10^{-24})(0.013)^2}{(2 \times 1.38 \times 10^{-23})(5.28 \times 10^{-7})} = 46.4$$

Rayleigh resolution (radians)

$$\theta = 1.22\,\frac{\lambda}{D}$$

λ = wavelength
D = diameter of objective

Example: The Hubble Space Telescope has a primary mirror that is 2.40 m in diameter. This telescope can, in theory, resolve two objects emitting light with a wavelength of 550 *nm* if these objects are separated by an angle of at least $2.80 \times 10^{-7}\,rad$. (In practice, a slightly poorer resolution is realized because of small imperfections in the optics.)

$$1.22\,\frac{(550 \times 10^{-9})}{2.4} = 2.80 \times 10^{-7}$$

Reduced mass

$$\mu = \frac{m_1 m_2}{m_1 + m_2}$$

$m_1, m_2 =$ masses

Example: The reduced mass of the hydrogen atom, which consists of one proton (p) and one electron (e) is approximately equal to the mass of the electron.

$$\mu = \frac{m_e m_p}{m_e + m_p} = m_e \left(\frac{1}{1 + \frac{m_e}{m_p}} \right) \approx m_e$$

Relativistic periastron precession

$$\dot{\omega} = \frac{6M\pi}{p}$$

$\dot{\omega}$ = Periastron precession in radians per revolution
M = Combined mass
p = semi-latus rectum, $= a(1 - e^2)$, where
a = semi-major axis
e = orbital eccentricity

Example: The binary pulsar PSR 1913+16 has a combined mass of 2.82787 solar masses. (In geometricized units, commonly used in general relativity calculations, the solar mass is given in metres as $1.476 \times 10^3\,m$.) The orbital semi-major axis is $1.9501 \times 10^9\,m$ and the orbital eccentricity is 0.6171395. Therefore, $p = 1.207382647 \times 10^9\,m$, and the resulting relativistic periastron precession is 4.2 degrees per year.

$$\dot{\omega} = \frac{6\,(2.82787 \times 1.476 \times 10^3)\,\pi}{1.207382647 \times 10^9} = 0.6516313823 \times 10^{-4}\,rad/rev$$

This is equivalent to 3.73357×10^{-3} degrees per revolution. The binary system makes 1130.8 revolutions in one year. Therefore, the total annual periastron precession of the binary system is

$$(3.73357 \times 10^{-3}) \times 1130.8 = 4.2\,deg/yr$$

Rising and setting times

$$\cos h = -\tan\delta\tan\phi + \frac{\sin a}{\cos\delta\cos\phi}$$

h = hour angle
δ = declination
a = altitude above horizon (= atmospheric refraction ~ -35’)
ϕ = latitude

Example: The right ascension and declination of Arcturus are, respectively, $\alpha = 14\,h\,15.7\,min$ and $\delta = 19^\circ\,11'$. The latitude of Vancouver is $49.2927^\circ\,N$. Arcturus rises in Vancouver at 6h 36min **local sidereal time** and sets at 21h 56min **local sidereal time**.

$$\cos h = -(\tan 0.3348123)(\tan 0.8603199) - \frac{\sin 0.010181}{(\cos 0.3348123)(\cos 0.8603199)} = -0.42091$$

$$\Longrightarrow h = 7\,h\;40\,min$$

Using the **local sidereal time** formula, subtract the hour angle (h) from the right ascension to get the sidereal rising time:

$$\Theta = \alpha - h = 14\,h\,16\,min - 7\,h\;\;40\,min = 6\,h\,36\,min$$

And add the two to get the sidereal setting time:

$$\Theta = \alpha - h = 14\,h\,16\,min + 7\,h\;\;40\,min = 21\,h\,56\,min$$

Roche's limit

$$r < f_R \left(\frac{\bar{\rho_p}}{\bar{\rho_m}}\right)^{\frac{1}{3}} R_p$$

f_R = constant, taken to be 2.456
$\bar{\rho_p}$ = average density of planet
$\bar{\rho_m}$ = average density of moon
R_p = radius of planet

Example: The average density of the planet Saturn is $687\,kg\,m^{-3}$ and its radius is $6.03 \times 10^7\,m$. A typical Saturnian moon has an average density of $1200\,kg\,m^{-3}$. Consequently, Saturn's Roche limit is $1.23 \times 10^8\,m$. Most of the Saturnian ring system lies within this distance, suggesting that the rings consist of debris left over from moons and asteroids that came too close to the planet.

$$r < 2.456\left(\frac{687}{1200}\right)^{\frac{1}{3}} 6.03 \times 10^7 = 1.23 \times 10^8$$

Rocket equation (Tsiolkovsky's equation)

$$\Delta v = v_e \ln\left(\frac{m_i}{m_f}\right)$$

Δv = change in velocity
v_e = velocity of exhaust
m_i = initial mass
m_f = final mass

Example: If a rocket ship drifting in space has a mass of 4000 *kg* and burns 3500 *kg* of fuel at a rate of $2 \times 10^3\, m\ s^{-1}$, its velocity will increase by 4158.9 *m/s*.

$$(2 \times 10^3) \ln\left(\frac{4000}{500}\right) = 4158.9$$

Root-mean-square speed of gas particles

$$v_{RMS} = \sqrt{\frac{3k_BT}{m}} = \sqrt{\frac{3RT}{M_m}}$$

k_B = Boltzmann constant
T = temperature
m = mass of particle or atom
R = universal gas constant
M_m = molar mass of particle or atom

Example: The root mean square (RMS) speed of an oxygen molecule at a temperature of 0 C (= 273 K) in the Earth's atmosphere is 461 m/s.

$$\sqrt{\frac{3 \times (8.314472\, J/mol\, K) \times 273\, K}{3.2 \times 10^{-2}\, kg/mol}} = 461\, m/s$$

Rosseland mean opacity

$$\frac{1}{\bar{\kappa}} \equiv \frac{\int_0^\infty \frac{1}{\kappa_\nu}\frac{\partial B_\nu(T)}{\partial T}d\nu}{\frac{\int_0^\infty \frac{\partial B_\nu(T)}{\partial T}d\nu}{\partial T}}$$

Instead of using this expression, it is usually easier to take the separate contributions to opacity and compare or average them. For instance, the formulas for bound-free, free-free, and electron scattering opacities are, respectively, as follows:

$$\bar{\kappa}_{bf} = 4.34 \times 10^{21}\,\frac{g_{bf}}{t}\,(1+X)\frac{\rho}{T^{3.5}}\,m^2\,kg^{-1}$$

$$\bar{\kappa}_{ff} = 3.68 \times 10^{18}\,g_{ff}\,(1-Z)(1+X)\frac{\rho}{T^{3.5}}m^2\,kg^{-1}$$

$$\bar{\kappa}_{es} = 0.02\,(1+X)\,m^2\,kg^{-1}$$

κ_ν = opacity at frequency ν
B_ν = **Planck's law**
T = temperature
g = Gaunt factor
t = guillotine factor (describes the contribution to opacity of an atom after it has been ionized)
X = mass fraction of hydrogen
Z = mass fraction of metals
ρ = density

Example: Using the following values, the bound-free, free-free, and electron scattering opacities at different depths within the Sun can be calculated:

$$\frac{g_{bf}}{t} = 0.1,\ g_{ff} = 1,\ X = 0.708,\ Z = 0.02$$

For $R = 0.1\,R_\odot$: $\rho = 85700\,kg\,m^{-3}$; $T = 1.3 \times 10^7\,K$

$$\bar{\kappa}_{bf} = (4.34 \times 10^{21})\,0.1\,(1 + 0.709)\frac{85700}{(1.3 \times 10^7)^{3.5}} = 0.16\ m^2\,kg^{-1}$$

$$\bar{\kappa}_{ff} = (3.68 \times 10^{18})\,1\,(1 - 0.02)(1 + 0.708)\frac{85700}{(1.3 \times 10^7)^{3.5}} = 0.067\,m^2\,kg^{-1}$$

$$\bar{\kappa}_{es} = 0.02\,(1 + 0.708) = 0.034\ m^2\,kg^{-1}$$

Similarly, for $R = 0.5\,R_\odot$: $\rho = 1000\,kg\,m^{-3}$; $T = 4 \times 10^6\,K$:
$\kappa_{bf} = 0.12, \kappa_{ff} = 0.05, \kappa_{es} = 0.034$

For $R = 0.9\,R_\odot$: $\rho = 25\,kg\,m^{-3}$; $T = 5.6 \times 10^5\,K$
$\kappa_{bf} = 2.8, \kappa_{ff} = 1.17, \kappa_{es} = 0.034$

It is clear that bound-free opacities dominate at every level within the Sun.

S

Scale factor

$$a = \frac{1}{1+z}$$

z = redshift

Example: Light from quasar PC 1247+346 was emitted at $z = 4.897$. At that time, the size of the Universe was about 17% of its current size.

$$\frac{1}{1+4.897} = 0.17$$

Schwarzschild radius

$$r_s = \frac{2\,G\,M}{c^2}$$

G = gravitational constant
M = mass of black hole
c = speed of light

Example: The black hole at the centre of the galaxy M 85 has a mass that is approximately 10^8 times that of the Sun. Its Schwarzschild radius is, therefore, $2.95 \times 10^{11}\,m$.

$$\frac{2\,(6.67408 \times 10^{-11})\,(10^8 \times 1.99 \times 10^{30})}{(2.998 \times 10^8)^2} = 2.95 \times 10^{11}$$

Signal-to-noise ratio

$$\frac{S}{N} = \frac{F A_\epsilon \tau}{\left(N_R^2 + \tau N_T\right)^{\frac{1}{2}}}$$

where

$$N_T = F A_\epsilon + i_{DC} + F_\beta A_\epsilon \Omega$$

where

$$A_\epsilon = A \epsilon Q_e$$

$F =$ point source signal flux on telescope (photons $s^{-1} m^{-2}$)
$A =$ telescope area (m^{-2})
$A_\epsilon =$ effective area
$N_R =$ readout noise (e^-)
$N_T =$ time dependent noise per unit time
$i_{DC} =$ dark current ($e^- s^{-1}$)
$F_\beta =$ background flux from sky ($photons\, s^{-1} m^{-2} arcsec^{-2}$)
$\Omega =$ pixel size (arcsec) (must be greater than seeing)
$\epsilon =$ telescope efficiency (dimensionless)
$\tau =$ integration time (s)

Example: When a CCD device with the specifications given below is used with a telescope with an efficiency of 0.5, a V filter, and an integration time of 100 seconds to view an object of magnitude 20 (F (flux) = 300 photons $s^{-1}\, m^{-2}$. See **flux: photon number**.) under ideal seeing conditions ($F_\beta =$ 100 $photons\, s^{-1}\, m^{-2}\, arcsec^{-2}$), the signal-to-noise ratio is 12.5.

Typical CCD specifications:

$A = 0.1\ m^{-2}$
$N_R = 12$
$i_{DC} = 1\ e^{-}\, s^{-1} pixel^{-1}$ at 35°
$\Omega = 4\ arcsec^2$

$$N_T = 300 \times 0.015 + 1 + 100 \times 0.015 \times 4 = 11.5$$

$$A_\epsilon = 0.1 \times 0.5 \times 0.3 = 0.015$$

$$\frac{S}{N} = \frac{300 \times 0.015 \times 100}{\left(12^2 + 100 \times 11.5\right)^{\frac{1}{2}}} = 12.5$$

Solar sail total force

$$F_{sail} = \frac{1}{2}\frac{R_{sail}^2 L_\odot}{cr^2} - \frac{GM_\odot m_{sail}}{r^2}$$

R_{sail} = radius of sail
$L_\odot$ = solar luminosity
c = speed of light
r = distance from Sun
G = gravitational constant
$M_\odot$ = solar mass
m_{sail} = mass of sail

Example: A 100% reflective solar sail with a radius of 600 m and a mass of 900 kg is located in the Earth's orbit but far enough from the Earth that the Earth's gravitational force on it is negligible. The sail points directly toward the Sun. The Sun's force on this sail is equal to the **light pressure** times the area, minus the **gravitational force** of the Sun, and amounts to 5 N.

$$\frac{1}{2}\frac{(600)^2\,(3.845\times10^{26})}{2.9979\times10^{8})\,(1.496\times10^{11})^2} - \frac{(6.67408\times10^{-11})\,(1.99\times10^{30})\,900}{(1.496\times10^{11})^2} = 5$$

Solid angle

$$d\Omega = \sin\theta\, d\theta\, d\phi$$

θ = zenith angle
ϕ = azimuthal angle

Example: The total solid angle about a point is equal to $4\pi\ sr$.

$$\oint d\Omega = \int_0^{2\pi}\int_0^{\pi}\sin\theta d\theta d\phi = 4\,\pi$$

Solid angle subtended by an ellipse

$$\Omega = \left(\frac{\pi}{4}\right)\left(\frac{a}{2}\right)\left(\frac{b}{2}\right)$$

a = major axis
b = minor axis

Example: The galaxy M31 subtends a major axis of 190.5 arc minutes and a minor axis of 60.96 arc minutes. Therefore, it subtends a solid angle of 2280.18 square arc minutes on the sky.

$$\Omega = \left(\frac{\pi}{4}\right)\left(\frac{190.5}{2}\right)\left(\frac{60.96}{2}\right) = 2280.18$$

Source function

$$S_\lambda \equiv \frac{j_\lambda}{\kappa_\lambda}$$

j_λ = emission coefficient ($W\,m^{-3}\,sr^{-1}\,Hz^{-1}$)
κ_λ = absorption coefficient (= opacity) (m^{-1})

Example: It can be shown that, for a plane-parallel, grey atmosphere in local thermodynamic equilibrium:

$$S = \langle I \rangle = \frac{3}{4\pi} F_{rad}\left(\tau_\nu + \frac{2}{3}\right)$$

where I = intensity; F_{rad} = the radiant flux, and τ = the optical depth. If $\tau = 2/3$, $F_{rad} = \pi S$. This is known as the Eddington-Barbier relation and states that the observed radiative flux from the surface of a star depends on the source function at an optical depth of 2/3.

$$S = \langle I \rangle = \frac{3}{4\pi} F_{rad}\left(\frac{2}{3} + \frac{2}{3}\right)$$

$$S = \frac{F_{rad}}{\pi}$$

$$F_{rad} = \pi S(\tau = 2/3)$$

Specific intensity: surface of star

$$I_\nu(t = 0, \nu) = \int_0^\infty S_\nu(t)\, e^{-\frac{t}{u}}\, \frac{dt}{u}$$

$S =$ **source function**
$\nu =$ frequency
$t =$ depth
$u = \cos\theta$
$\theta =$ angle between direction of ray and normal to the surface

Example: A stellar **source function** may often be represented by $S(t) = a_\nu + b_\nu\,\tau^n$, where n is a real number. Consider a case where n = 1. The specific intensity at the surface of the star (u > 0) is then $a_\nu + b_\nu\,u$. (A more detailed calculation of a similar problem may be found under "Specific intensity" in Part 2.)

$$I_\nu(t = 0,\nu) = \int_0^\infty (a_\nu + b_\nu\,t)\,e^{-\frac{t}{u}}\,\frac{dt}{u}$$

$$= \int_0^\infty a\,e^{-\frac{t}{u}}\,\frac{dt}{u} + \int_0^\infty b\,t\,e^{-\frac{t}{u}}\,\frac{dt}{u}$$

$$= \frac{a}{u}\left[-ue^{-\frac{t}{u}}\right]_0^\infty + \frac{b}{u}\left[-ue^{-\frac{t}{u}}(t+u)\right]_0^\infty = a_\nu + b_\nu u$$

Speed of accelerated object $v_{x,final} = v_{x,initial} + a_x\,t$

Example: If a space ship moving in a straight line at 1000 m/s is accelerated at a rate of 0.3 m/s/s for four hours (14400 s), its velocity will increase to 5320 m/s.

$$1000 + 0.3\,(14400) = 5320$$

Spiral galaxy radius-luminosity relation for Sa-Sc types $\log_{10} R_{25} = -\,0.249\,M_B - 4.00$

$R_{25} =$ disk radius in kpc corresponding to a 25 B-mag/arcsec^2 surface brightness
$M_B =$ absolute B-magnitude

Example: The Sc galaxy M101 has an absolute B-magnitude of -21.51. From this, a radius of 22.7 kpc can be calculated.

$$log_{10}R_{25} = -0.249\,(-21.51) - 4.00 = 22.7$$

Stellar Lifetime: Main Sequence star

$$t = (1 \times 10^{10})\,\frac{M}{L}$$

$t =$ time in years
$M =$ mass
$L =$ luminosity

Example: A B2 star has an absolute magnitude of -2.5, a bolometric correction of -2.2, and a mass of approximately 8.3 $M_{\odot}$. Therefore, its estimated lifetime is 1.4×10^7 years.

First, use the **bolometric correction** formula to find the absolute bolometric magnitude:

$$M_{bol} = -2.5 - 2.2 = -4.7$$

Next, use the **bolometric equation** to find the ratio of the star's luminosity to that of the Sun (+4.74):

$$-4.7 - 4.74 - = -2.5\ log\left(\frac{L}{L_{\odot}}\right) \Longrightarrow \frac{L}{L_{\odot}} = 5970$$

Therefore, the lifetime of this star is estimated to be:

$$t = (1 \times 10^{10})\frac{8.3}{5970} = 1.4 \times 10^{7}$$

Synchrotron power

$$P = \frac{4}{3}\sigma_T\, c\, \beta^2\ \gamma^2 \frac{B^2}{8\,\pi}\, 10^{15}\left[\frac{s^2\, A^2}{m\, kg}\right]$$

where

$$\gamma = \sqrt{\left(2.222372826 \times 10^{-13}\left[\frac{s}{m}\right]\right) n\ \left(\frac{m_e\, c}{e\, B}\right)}$$

σ_T = Thomson cross-section
c = speed of light
β = velocity (v) divided by the speed of light
γ = **Lorentz factor** $\left(\gamma = \frac{1}{\sqrt{1-\beta^2}}\right)$
B = magnetic field strength
n = peak frequency
m_e = mass of electron
e = electron charge

Example: A relativistic synchrotron-powered electron at a peak frequency of $9.109 \times 10^{-28}\,Hz$, moving in a magnetic field of $1.43 \times 10^{-12}\,T$, radiates a power of $2.753 \times 10^{-15}\,W$.

First, calculate the **Lorentz factor** for a particle in a magnetic field:

$$\gamma = \sqrt{\left(2.222372826 \times 10^{-13}\right)\ (4.8 \times 10^{18})\ \left(\frac{(9.109 \times 10^{-31})\,(3 \times 10^{8})}{(1.602 \times 10^{-19})\,(1.43 \times 10^{-12})}\right)} = 3.567 \times 10^{7}$$

With a **Lorentz factor** of this magnitude, β is almost exactly equal to 1 and may be removed from the power equation.

$$P = \frac{4}{3}(6.6524 \times 10^{-29})\,(3 \times 10^{8})\,(3.567 \times 10^{7})^{2}\left(\frac{(143.0 \times 10^{-12})^{2}}{8\,\pi}\right)\,10^{15} = 2.755 \times 10^{-15}\,W$$

Synodic revolution period

For inferior planets:

$$\frac{1}{S} = \frac{1}{P} - \frac{1}{P_{\oplus}}$$

For superior planets:

$$\frac{1}{S} = \frac{1}{P_{\oplus}} - \frac{1}{P}$$

P = period of planet's revolution
$P_{\oplus}$ = period of Earth's revolution (365.356 days)

Example: Venus revolves around the Sun in 224.701 days. Therefore, its synodic revolution period is 583.92 days.

$$\frac{1}{S} = \frac{1}{224.701} - \frac{1}{365.256} \Longrightarrow S = 583.92$$

T

Thermal energy

$$E_{thermal} = \frac{1}{2}mv^2 = \frac{3}{2}kT$$

m = mass
v = velocity
k = Boltzmann constant
T = temperature

Example: At the surface of the Sun, any molecule, regardless of species, has a thermal energy of 1.197 J.

$$\frac{3}{2}(1.38 \times 10^{-23})\,5778 = 1.197$$

U

Universal law of gravity

$$F_g = \frac{Gm_1m_2}{r^2}$$

F_g = Gravitational force
m = masses
r = distance between masses

Example: The mass of the asteroid 243 Ida is approximately $4.2 \times 10^{16}\, kg$. The mass of its moon Dactyl is approximately $3.6 \times 10^{13}\, kg$. The distance between them is about 90000 m. Therefore, the gravitational force between them is approximately $1.25 \times 10^{10}\, N$.

$$\frac{(6.67408 \times 10^{-11})(4.2 \times 10^{16})(3.6 \times 10^{13})}{(90000)^2} = 1.25 \times 10^{10}$$

V

Velocity dispersion

$$\sigma = \left(\frac{3}{5}\frac{GM}{R}\right)^{\frac{1}{2}}$$

G = gravitational constant
M = mass
R = radius

Example: A proto-galactic nebula of fully ionized gas with a mass of 5×10^{11} solar masses and a radius of 50 kpc would exhibit a velocity dispersion of approximately 160,000 m/s.

$$\left(\frac{3}{5}\frac{(6.67408 \times 10^{-11})(5 \times 10^{11})(1.99 \times 10^{30})}{(50 \times 10^{3})(3.086 \times 10^{16})}\right)^{\frac{1}{2}} = 160{,}694$$

Velocity in a bound orbit

$$v^2 = G(m_1 + m_2)\left(\frac{2}{r} - \frac{1}{a}\right)$$

v = velocity
G = gravitational constant
m = masses of orbiting bodies
r = distance between the two bodies
a = semi-minor axis

Example: The orbital speed of Halley's comet at the semi-minor axis of its orbit is 7.0 km/s. Note that the mass of the comet is insignificant compared with the mass of the Sun ($1.99 \times 10^{30}\,kg$), and that *r* = *a* at this point in the orbit. (*a* = $2.68 \times 10^{12}\,m$.)

$$\sqrt{\frac{(6.67408 \times 10^{-11}) \times (1.99 \times 10^{30})}{2.68 \times 10^{12}}} = 7043\,m/s$$

Virial temperature

$$T_{virial} = \frac{\mu\, m_H\, \sigma^2}{3k}$$

μ = **mean molecular weight**
m_H = mass of hydrogen atom
σ = **velocity dispersion**
k = Boltzmann constant

Example: A proto-galactic nebula of fully ionized gas consisting of 90% oxygen and 10% helium (X = 0.7; Y = 0.3) has a **mean molecular weight** of 0.6 at full ionization. If the nebula has a **velocity dispersion** of approximately 160 km/s, its virial temperature, based on the **virial theorem** is $6.2 \times 10^5 K$.

$$\frac{0.6\,(1.6727 \times 10^{-27})\,(160000)^2}{3\,(1.38 \times 10^{-23})} = 6.2 \times 10^5$$

Virial Theorem

$$\langle KE \rangle = -\frac{1}{2}\langle U \rangle$$

KE = kinetic energy
U = potential energy

Example: The **gravitational potential energy** of the Sun is $-2.82 \times 10^{41}\,J$. The **kinetic energy** of its gas is $1.41 \times 10^{41}\,J$.

$$\langle 1.41 \times 10^{41} \rangle = -\frac{1}{2}\langle -2.82 \times 10^{41} \rangle$$

W

Width of opening angle

$$\theta = \frac{1}{\gamma}$$

γ = **Lorentz factor**

Example: A relativistic opening angle of 1.6° (0.27925 rad) has been measured for the synchrotron radiation of the elliptical radio galaxy Cygnus A. This indicates particle speeds with a **Lorentz factor** equal to 35.8, corresponding to a speed of 0.999 times the speed of light.

$$0.027925 = \frac{1}{\gamma} \Longrightarrow \gamma = 35.8$$

Wien's approximation
$h\nu/kT > 2.3$

$$B_\lambda(T) = \frac{2hc^2}{\lambda^5} e^\beta$$

where $\beta = \dfrac{hc}{k\lambda T}$

h = Planck's constant
c = speed of light
λ = wavelength
k = Boltzmann's constant
T = temperature

Example: (This formula is an approximation of **Planck's law** for the short-wavelength side of the spectrum.) For a wavelength of $10^{-5}\,m$ and a temperature of 213 K, the value of β is 6.74, leading to a Wien approximation of $1.39 \times 10^{6}\,J\,m^{-3}\,s^{-1}\,sr^{-1}$. (The more accurate **Planck's law** value is $1.33 \times 10^{6}\,J\,m^{-3}\,s^{-1}\,sr^{-1}$).

$$\beta = \frac{(6.26068 \times 10^{-34}\,J\,s) \times (2.997925 \times 10^{8}\,m\,s^{-1})}{(1.3066 \times 10^{-23}\,J\,K^{-1}) \times 10^{-5}\,m \times 213\,K} = 6.74$$

$$\frac{2\,(6.26068 \times 10^{-34}\,J\,s)\,(2.997925 \times 10^{8}\,m\,s^{-1})^{2}}{(10^{-5}\,m)^{5}}\,e^{-\beta} = 1.39 \times 10^{6}\,J\,m^{-3}\,s^{-1}\,sr^{-1}$$

Wien's displacement law

$$\lambda_{peak}\,T = 2.898 \times 10^{-3}\,m\,K$$

λ_{peak} = peak wavelength emitted at a given temperature
T = temperature

Example: A nuclear bomb produces a temperature of roughly 10^{7} degrees. Consequently, the radiation it emits has a wavelength around $2.9 \times 10^{-10}\,m$. This is in the X-ray range.

$$\lambda = \frac{2.898 \times 10^{-3}}{10^{7}} = 2.9 \times 10^{-10}$$

Work equation

$$W = \int_{C} f\,dt$$

f = force function
t = distance along a trajectory

Example: The work done by gravity on an object of mass m in a circular orbit around an object of mass M is zero.

The force function here is **the Universal law of gravity:**

$$\overrightarrow{F} = -\frac{GMm}{|\vec{r}|^2}\hat{r}$$

Vector equation of a circle:

$$\vec{r} = R\cos(t)\hat{i} + R\sin(t)\hat{j}$$

$$\hat{r} = \frac{\vec{r}}{|\vec{r}|} = \frac{\vec{r} = R\cos(t)\hat{i} + R\sin(t)\hat{j}}{R} = \cos(t)\hat{i} + \sin(t)\hat{j}$$

$$\frac{dr}{dt} = -R\sin(t)\hat{i} + R\cos(t)\hat{j}$$

$$W = -\frac{GMm}{R^2}\int_0^{2\pi}(\cos(t)\hat{i} + \sin(t)\hat{j})(-R\sin(t)\hat{i} + R\cos(t)\hat{j})dt = 0$$

Part II

An Alphabetical List of Extended Astrophysical Examples with Solutions, Using Maple™